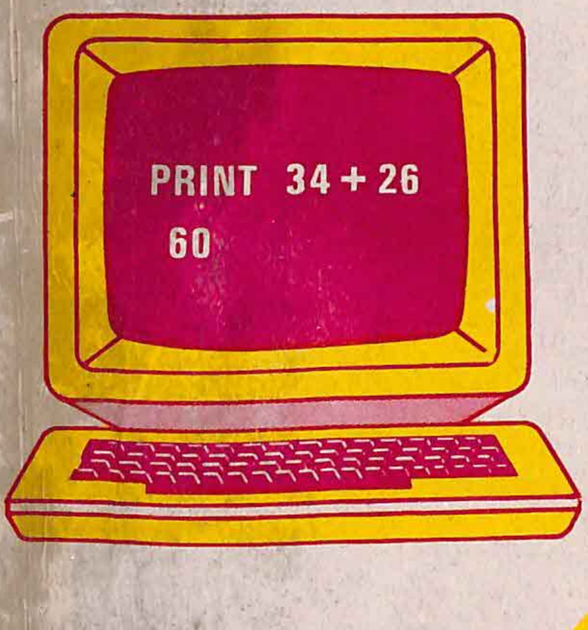
FUNDAMENTALS OF MATHEMATICS

VOL. II

R. P. GOEL



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Rased on the latest syllabus prescribed by the Central Board of Secondary Education, New Delhi under the New Education Policy

FUNDAMENTALS OF MATHEMATICS

Vol. II

(FOR CLASS X)

2000

RAJENDRA PRASAD GOEL
Formerly, Senior Mathematics Teacher,
St. Xavier's School,
Delhi.

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PREFACE

This book has been written for the students of Class X of Secondary Schools affiliated to the Central Board of Secondary Education (CBSE), New Delhi, according to new syllabus developed by National Council of Educational Research and Training (NCERT), New Delhi, under the National Policy of Education.

In fact, it is an effort made to cater to the needs of the average students and the brighter students as well who have to learn Mathematics as a compulsory subject. Learning of Mathematics is essential for them to study other subjects as an integrated course and to face multi-dimensional and challenging situations in days to come. To make learning of the subject interesting and meaningful to the day-to-day needs of would be citizens, I have presented the contents in a new format which is based on my long experience of teaching the subject and varied writing experience of text-books on Mathematics.

The following are the special features of the book :

1. The emphasis has been laid on the clear understanding of basic concepts and principles by the students.
2. Treatment of the subject matter is simple and logical. The matter has been developed successively with the help of previous knowledge of the students.
3. Each chapter comprises several functional units and each unit is designed to cover one or two teaching periods.
4. Numerous worked-out examples have been provided for better understanding of the contents by the students and to explain the methodical working of the problems fully.
5. There are sufficient well-graded questions related to our daily life in each exercise. Each exercise has been divided into three Sections A, B and C. Section A consists of objective or short-answer questions, while Section B has simple applications and Section C has more difficult questions.
6. After each chapter a review exercise has been added, enriched by the inclusion of questions set in public examinations for effective revision side by side.
7. Working rules, important results and points worth remembering and deductions are given at appropriate places.
8. Wherever necessary hints have been provided to help students to solve difficult questions with confidence.

9. The figures, diagrams and graphs given in the book are complete, neatly drawn and self-explanatory.

10. Only agreed conventions, popular symbols or notations and metric units have been used throughout the book.

11. Answers to questions have been carefully checked for their correctness and accuracy.

I am indebted to Dr. Vijay Bhushan Aggarwal, Founder Head of the Deptt. of Computer Science, Delhi University for writing Chapter 12 on Computing II in the book.

I would like to request my fellow teachers and pupils to send their constructive criticism so that the book may be improved in its future editions.

Ashok Vihar, Delhi

RAJENDRA PRASAD GOEL

SYLLABUS IN MATHEMATICS FOR CLASS X PRESCRIBED BY CBSE

One Paper — 3 Hours — Marks : 100

UNITS		Marks			Marks
1.	Algebra	30	4.	Geometry	30
2.	Arithmetic and Mensuration	10	5.	Statistics	08
3.	Trigonometry	12	6.	Computing (II)	10

Unit I : Algebra

Linear equatians in two variables. Linear equations in two variables and its graph. System of two linear equations in two variables, Solution of the system of equations by graphical method. Consistency/inconsistency of the equations.

Algebraic method of the solution of a system of equations. Applications involving the system of equations from different areas.

Rational Expressions. Meaning of a rational expression, addition, subtraction, multiplication of rational expressions. Factorization of expressions involving cyclic factors, Ratio and proportion, Componendo, dividendo, alternendo, invertendo, etc , and their application.

Quadratic Equation. Meaning and standard form of a quadratic equation $ax^2+bx+c=0$, $a\neq 0$

Solution of $ax^2+bx+c=0$, $a\neq 0$ (*i*) by factorization (*ii*) by quadratic formula. Discriminant of the quadratic equation and nature of the roots. Applications involving quadratic equation and nature of the roots.

Applications involving quadratic equation from several areas.

Solution of equations reducible to quadratic form. Factorization of quadratic polynomials by using quadratic formula (when other methods are not easily applicable)

Unit II : Arithmetic and Mensuration

Area and Volume. Review of concepts such as area of a circle, sector, segment ; surface areas and volumes of cubes, cuboids, cones, cylinders, spheres ; area of four walls of a room studied in earlier classes and solution of problems (of higher difficulty level), using logarithmic tables for computational work.

Unit III : Trigonometry (20 Periods)

Trigonometrical Identities. $\sin^2 A+\cos^2 A=1$; $\sec^2 A=1+\tan^2 A$; $\text{cosec}^2 A=1+\cot^2 A$

Proving simple identities based upon the above.

Trigonometrical ratios of complementary angles

$\sin(90°-A)=\cos A$, $\text{cosec}(90°-A)=\sec A$, $\cos(90°-A)=\sin A$, $\sec(90°-A)=\text{cosec}\, A$, $\tan(90°-A)=\cot A$, $\cot(90°-A)=\tan A$. Simple problems based upon the above

Heights and Distances. Reading of trigonometrical tables. Solution of simple problems on heights and distances, using trigonometrical tables and logarithmic tables.

Unit IV : Geometry

A number of propositions in Geometry are listed below. Most of them have already been learnt at the Upper Primary stage by verification/experiments. At the Secondary stage the purpose is to acquaint the pupil with the nature and method of a geometrical proof. In order to see that the burden on the pupil is not much it may not be necessary to give proofs for all the propositions. So, a few of these may be selected in such a way that they reflect types of proof like direct proof, proof by contradiction, proof by exhaustion, proofs using various criteria like SAS, SSS etc. But there should be a large number of exercises where the pupil will be required to prove riders, applying the knowledge and understanding of the various theorems, so that the pupil develops the ability of identifying the inter-relationship between different parts of the problems and draws conclusions through reasoning, which is one of the main objectives of teaching Mathematics in general and Geometry in particular.

Similar Triangles

*1. If a line is drawn parallel to one side of a triangle, the other two sides are divided in the same ratio.

2. If a line divides any two sides of a triangle in the same ratio, the line is parallel to the third side.

3. If in two triangles, the corresponding angles are equal (*i.e.,* if the two triangles are equiangular), their corresponding sides are proportional (Axiom).

4. If the sides of two triangles are proportional the triangles are equiangular (Axiom).

5. If corresponding angles of two triangles are equal, then the triangles are similar (Axiom).

6. If corresponding sides of two triangles are proportional, then the triangles are similar (Axiom).

7. If one angle of a triangle is equal to one angle of the other and the sides including these angles are proportional, the triangles are similar (Axiom).

8. If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to each other.

*9. The ratio of the areas of similar triangles is equal to the ratio of the squares on the corresponding sides.

*10. In a right triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

*11. In a triangle, if the square on one side is equal to the sum of the squares on the remaining two, the angle opposite the first side is a right angle.

Circles. 1. Two circles are congruent if and only if they have equal radii.

2. If the areas of a cîrcle are congruent. their corresponding chords are equal and its converse.

3. A perpendicular from the centre of a circle to a chord bisects the chord and conversely, the line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.

4. There is one and only one circle passing through three given non-collinear points.

5. Equal chords of a circle (or of congruent circles) are equidistant from the centres and conversely, chords of a circle (or of congruent circles) that are equidistant from the centres are equal.

*6. The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.

*7. The angle in a semi-circle is a right angle and its converse.

*8. Angles in the same segment of a circle are equal.

9. If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the segment, the four points lie on the same circle.

10. Equal chords subtend equal angles at the centre and conversely, if the angles subtended by the chords at the centre (of a circle) are equal, then the chords are equal.

11. Two arcs of a circle are congruent if the angles subtended by them at the centre are equal and its converse.

*12. The sum of the opposite angles of either pair of a cyclic quadrilateral is 180° and conversely, if a pair of opposite angles of a quadrilateral are supplementary then the quadrilateral is cyclic (proof of converse not required).

13. A tangent at any point of a circle is perpendicular to the radius through the point of contact.

14. The lengths of two tangents from an external point to a circle are equal.

15. If two chords of a circle intersect inside or outside the circle then the rectangle formed by the two parts of one chord is equal in area to the rectangle formed by the two parts of the other.

*16. If PAB is a secant to a circle intersecting the circle at A and B and PT is a tangent, then $PA \times PB = PT^2$.

*17. If a line touches a circle and from the point of contact a chord is drawn, the angles which this chord makes with the given line are equal respectively to the angles formed in the corresponding alternate segments, and the converse.

18. If two circles touch each other, the point of contact lies on the line joining their centres.

Note : Proofs of Theorems starred (*) are to be done.

Constructions. 1. Construction of a circum-circle and an incircle of a triangle. 2. Construction of a triangle, given base, vertical angle and either altitude or median through vertex. 3. Construction of a cyclic quadrilateral with one vertex angle as a right angle. 4. Construction of figures (triangles, quadrilaterals, etc.) similar to the given figures as per the given scale factor.

Unit IV : Statistics

Mean of grouped data, Median of ungrouped data.

Descriptive explanation of mortality tables, cost of living index, price index. etc.

Unit V : Computing (II)

Flow charts involving loops—algorithms for mathematical problems already studied from topics such as profit and loss, ratio and proportion, simple and compound interest, discount ; HCF and LCM etc.—Easy exercises.

CONTENTS

1

LINEAR EQUATIONS IN TWO VARIABLES

1·1. LINEAR EQUATIONS IN TWO VARIABLES

You have learnt about linear equations in one variable in your previous class. A linear equation in one variable is of the form $ax+b=0$, where a and b are real numbers and $a\neq 0$. Here a is the *co-efficient* of x and b is the *constant term*. You have also learnt how to solve such equations. The solution of $ax+b=0$ is $x=-\frac{b}{a}$. We also say that $-\frac{b}{a}$ is the *root* of the equation.

Consider the following equations :

$$x-y=3, \qquad 3x+5y=9, \qquad \frac{3}{8}x=\frac{5}{6}-\frac{3}{5}y$$

Each of these equations contains two variables x and y and real numbers. These are linear equations in two variables over R.

An equation in two variables x, y is said to be linear, if it is of the form $ax+by+c=0$, where $a, b, c \in R$ and $a\neq 0$, $b\neq 0$.

Here a is called the **co-efficient** of x, b is called the **co-efficient** of y and c is the *constant term*.

Consider the equation given below :

$$x+y=5 \qquad \text{[Domain of the variables is } R\text{]}$$

When we assign any value to x, we find a unique value of y.

If $x=1$, then $y=4$. So ordered pair (1, 4) makes the equation true.

If $x=2$, then $y=3$, the ordered pair (2, 3) also makes the above equation true.

If $x=0$, then $y=5$, the ordered pair (0, 5) also makes the given equation true.

We say that the ordered pairs (1, 4), (2, 3) and (0, 5) are **solutions** of the given equation.

Are these the only solutions of the given equation ?

Other such ordered pairs are (3, 2), (4, 1), (5, 0), (0, 5). You can go on assigning real values to one variable. In each case you will be getting the corresponding value for the other variable such that the ordered pairs so obtained make the equation *true*.

The set of these ordered pairs in an *infinite* set. Each ordered pair corresponds to a point on a *real number plane*.

Let us consider the equation $x+y=7$.

This is a linear equation in x and y over R.

By giving real values to x, we get the corresponding values of y which make the equation true.

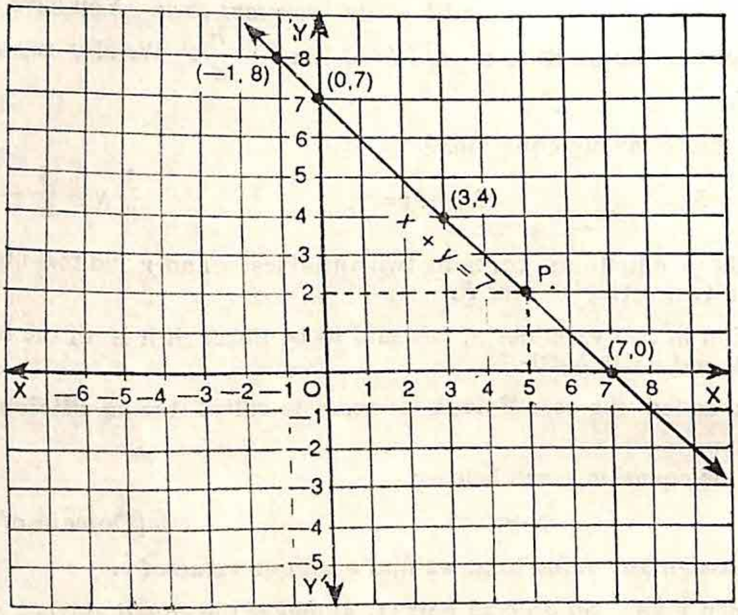
Thus, there are infinite number of ordered pairs which are the solutions of the given equation.

Let us write some of these in the form of a table, as given below :

$x=$	0	3	7	-1	-3
$y=$	7	4	0	8	10

Let us plot all those points whose coordinates are these ordered pairs.

What do you observe ? Are these points collinear ?



By plotting these points, we find that they all lie on a line.

We say that the graph of the linear equation $x+y=7$ is a *line*. The coordinates of any point on the line will satisfy this equation. Let us take a point P on the line whose coordinates are (5, 2). Now $x=5$ and $y=2$ satisfy the equation $x+y=7$. We, thus, see that any point on the graph of $x+y=7$ gives us a solution of the equation. Note that it is true for every linear equation in two variables over R.

Hence, the graph of $ax+by+c=0$ is a **line** and every point on the graph of $ax+by+c=0$ gives a **solution** of the equation.

Although we need to plot only two points to determine the graph of a linear equation over the real numbers, it is a good practice to plot a third point as a check.

Example 1. Draw the graph of the solution set of the equation

$$2x-3y+12=0, \quad x, y \in R.$$

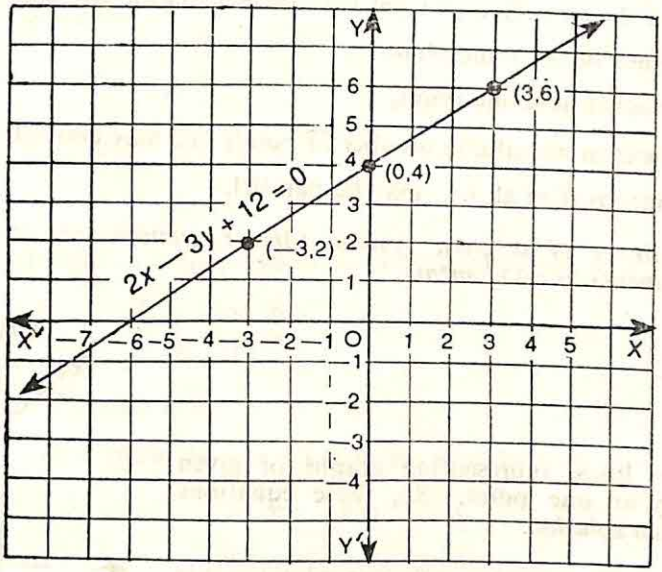
Solution. $2x-3y+12=0$

or $-3y=-2x-12$ or $3y=2x+12$

$\therefore$ $y=\dfrac{2x+12}{3}$

$y=\frac{2x+12}{3}$	$x=$	0	3	-3
	$y=$	4	6	2

Let us plot the ordered pairs (0, 4), (3, 6) and (−3, 2).



By joining these points, we get a line which is the required graph of the equation $2x-3y+12=0$.

Note that *the solution set consists of all the points on the line.*

In case of *fractional numbers* in ordered pairs, choose a suitable scale for plotting the points.

EXERCISE 1 (a)

(Section A)

Draw the graph of the solution set of each of the following equations :

1. $x=4$.
2. $y=-5$.
3. $x+y=0$.
4. $x+8=0$.
5. $y=9$.
6. $x-y=0$.
7. $x+y=6$.
8. $x+y+7=0$.
9. $x=2y$.

(Section B)

Draw the graphs of the following equations :

10. $4x-y-5=0$.
11. $3x+5y=15$.
12. $2y-x=6$.
13. $5x-3y=10$.
14. $\frac{x}{4}-\frac{y}{3}=1$.
15. $4x+7y+28=0$.

1·2. SYSTEM OF EQUATIONS

A pair of linear equations in two variables can be compounded by the connective '*and*'. Then the compounded equations form a **system of linear equations in two variables.**

A system of equations is usually written as :

$$2x-y=7 \text{ and } 3x-2y=9 \; ; \qquad \begin{cases} 2x-y=7 \\ 3x-2y=9 \end{cases}$$

The truth set of the system will consist of the *common solutions* of the constituent equations *i.e.*, the truth set of the system is the *intersection* of the truth sets of the constituent equations.

The graph of a linear equation in two variables is a line. When we draw graphs of two such equations on the same axes, we get two distinct lines in the same cartesian plane.

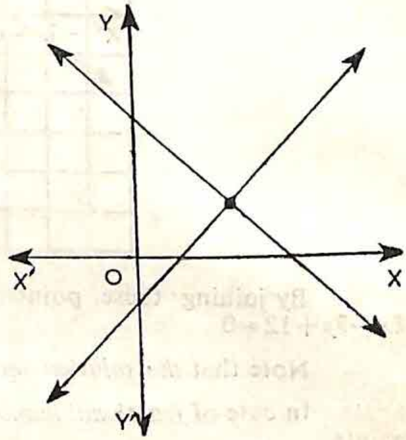
Any pair of lines in the same plane

(*i*) may intersect in just one point,

(*ii*) may intersect in an infinite number of points *i.e.*, may coincide,

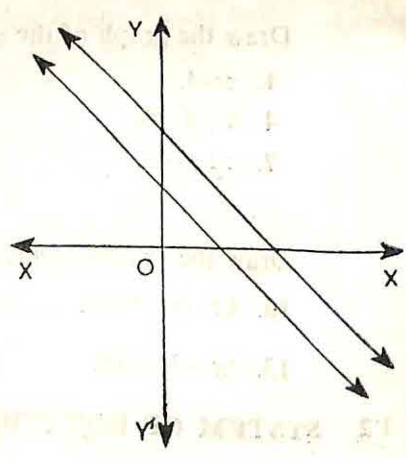
(*iii*) may not intersect at all *i.e.*, may be parallel.

Thus, *the truth set of a given system of linear equations may contain one element or an infinite number of elements or no elements.*



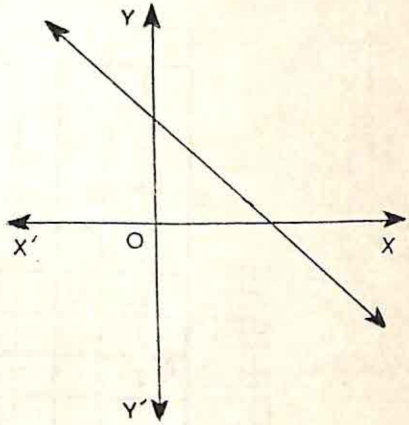
Here the two lines, representing graphs of given equations meet only at one point. So, these equations have *only one common solution.*

If a system of equations has only *one* solution, it is called **consistent.**



Here the two lines, representing graphs of given equations do not intersect at all *i.e.*, they are parallel. So, these equations have *no common solution.*

If the system of equations has *no* solution, it is called **inconsistent**.



Here the two lines, representing graphs of given equations coincide with each other. So, these equations have *infinite number of common solutions.*

If a system of equations has an *infinite* number of solutions, it is called **dependent.**

Example 2. Graph the solution set of the system of equations :

$$\begin{cases} 4x-3y=5 \\ 4x-3y=9 \end{cases}$$

State which type of system it is.

Solution. (1) Let us draw the graph of the first equation.

$$4x-3y=5 \qquad \text{or} \qquad 4x-5=3y$$

$$\text{Then} \qquad y=\frac{4x-5}{3}$$

$x=$	2	5	-1
$y=$	1	5	-3

On plotting the points (2, 1), (5, 5) and $(-1, -3)$, we get the line p.

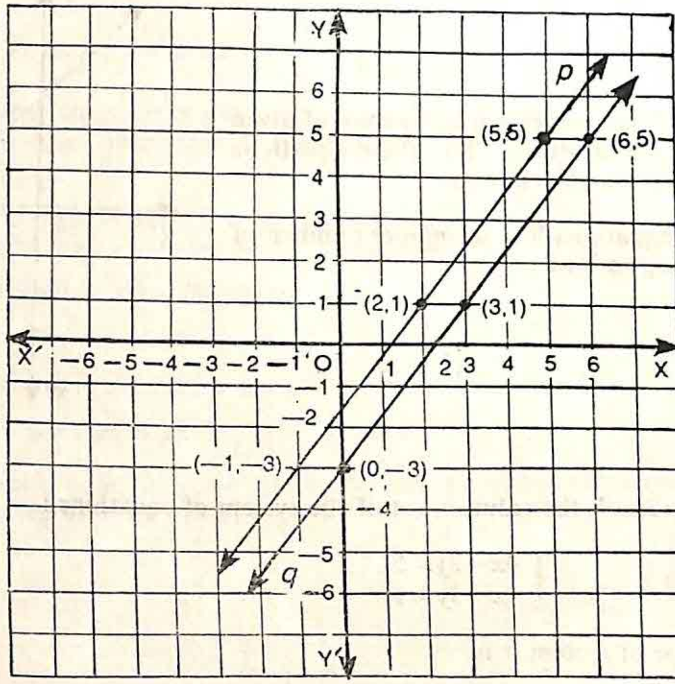
(2) Let us draw the graph of the second equation on the *same* axes.

$$4x-3y=9 \qquad \text{or} \qquad 4x-9=3y$$

$$\text{Then} \qquad y=\frac{4x-9}{3}$$

$x=$	0	3	6
$y=$	-3	1	5

On plotting the points $(0, -3)$, $(3, 1)$ and $(6, 5)$, we get the line q.



Scale—1 division=1 unit

(3) We note that lines p and q are *parallel*. So, there is *no* solution for the system of equations.

The system of equations is **inconsistent**.

Example 3. Graph the solution of the system of equations

$$4x-3y=5$$
$$x+2y=4$$

State which type of system it is.

Solution. (1) Taking the first equation,

$$4x-3y=5 \quad \text{or} \quad 3y=4x-5 \quad \text{or} \quad y=\frac{4x-5}{3}$$

$x=$	2	-1	0
$y=\dfrac{4x-5}{3}$	1	-3	$-\dfrac{5}{3}$

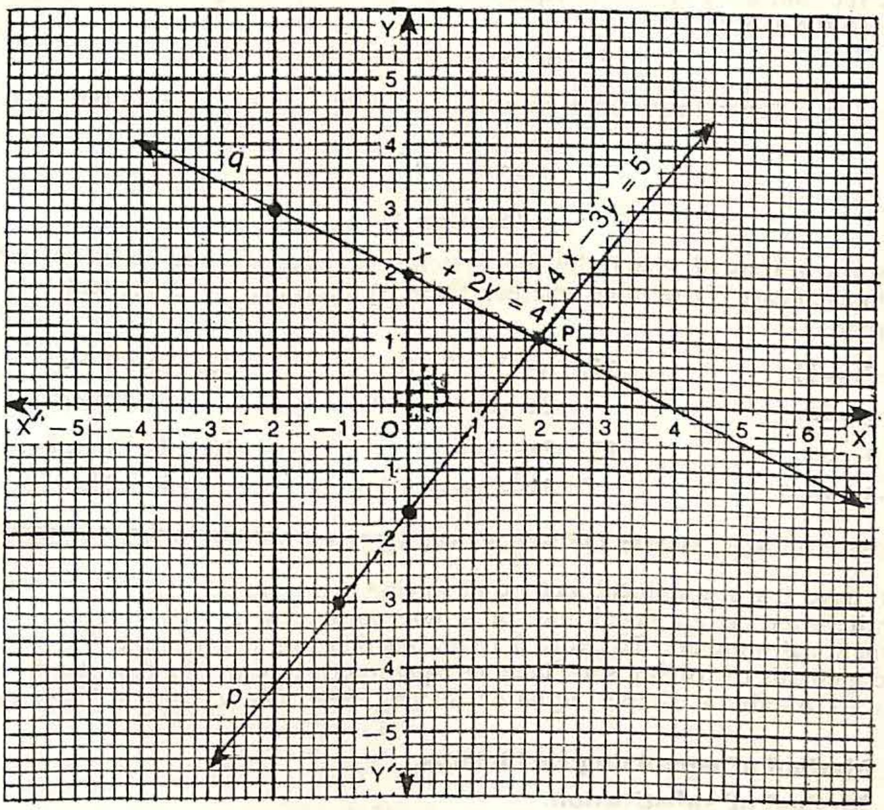
On plotting the points $(2, 1)$, $(-1, -3)$, $(0, -\frac{5}{3})$, and joining them, we get line p.

Taking the second equation, $x+2y=4$

or $\quad 2y=4-x \qquad$ or $\quad y=\dfrac{4-x}{2}$

$x=$	0	-2	2
$y=\frac{4-x}{2}$	2	3	1

Plotting the points (0, 2), (−2, 3), (2, 1) and joining them, we get line q.



Scale—5 divisions = 1 unit

The graphs of two equations *i.e.*, lines p and q intersect each other at the point P only whose coordinates are (2, 1). So, the solution of the system is $x=2$, $y=1$.

Hence, the system of equations is *consistent*.

EXERCISE 1 (b)

(Section A)

Solve the following systems of linear equations graphically :

1. $x+y=5$
   $x-y=3$.
2. $x-y=3$
   $x+y=1$.
3. $2x+y=12$
   $x+y=10$.

(Section B)

Solve each of the following systems of linear equations by drawing their graphs :

4. $2x+3y=8$
   $x+2y=5$.
5. $2x-3y=12$
   $2x+3y=24$.
6. $y-x+2=0$
   $x-2y-4=0$.

7. Solve the following equations graphically :

$$3x+2y=6 \text{ and } 4y-5x=1,$$

taking 2 *cm* to represent one unit on both the axes.

8. Taking 1 *cm* to represent 1 unit on each axis, draw graphs of the equations

$$x+2y=7 \text{ and } y-3x=1$$

and find their solution.

9. Solve the following system of equations graphically, taking 2 *cm* to represent one unit on both the axes :

$$3x-y+4=0;\ 5x+y+4=0.$$

10. Solve the following equations graphically, taking 2 *cm*=1 unit on each axis :

$$4x+3y=5$$
$$x-2y=-7.$$

(Section C)

Draw the graphs of the following systems of simultaneous equations and find the solution set in each case. Determine whether these systems are (*a*) consistent, (*b*) dependent, (*c*) inconsistent.

11. $5x-3y=11$, $3x+y=15.$

12. $x+3y-7=0$, $2x+6y-14=0.$

13. $2x=8+3y$, $4x=6y+5.$

14. $3y-2x=7$, $5x+3y=-7.$

15. $2x+3y=13$, $5x-2y=4.$

16. $4x-y-31=0$, $5x-24y-16=0.$

1·3. ALGEBRAIC METHODS OF SOLVING SIMULTANEOUS EQUATIONS

You have learnt graphical method of solving two simultaneous linear equations in two variables. Quite often the graphical method is not convenient specially when coordinates of points are fractional numbers. To avoid inaccuracies that can occur in the drawing of graphs, the algebraic methods of solving simultaneous equations are used. The first step in solving these equations is to obtain a simple equation having only *one variable*. This can be done in three ways :

(1) **Method of equalizing coefficients.**

(2) **Method of substitution.**

(3) **Method of comparison.**

Example 4. Solve

$$4x+3y=25$$
$$7x+8y=52$$

Solution. Let us name the equations,

$$4x+3y=25 \quad ...(1)$$
$$7x+8y=52 \quad ...(2)$$

Here the coefficients of neither x nor y in the two equations are the same.

We multiply both sides of equation (1) by 7 and equation (2) by 4.

$$28x+21y=175 \quad ...(3)$$
$$28x+32y=208 \quad ...(4)$$

Now the numerical coefficients of x in equations (3) and (4) are the *same*.

We subtract (3) from (4) to eliminate x.

$$11y=33 \qquad \therefore \quad y=3$$

To find x, we substitute $y=3$ in (1).

$$4x+3\times3=25 \quad \text{or} \quad 4x=25-9$$

$$\text{or} \quad 4x=16 \quad \therefore \quad x=4$$

So, $x=4, y=3$

Hence, the solution set is $\{(4, 3)\}$.

Note that the system is **consistent**.

This method of **elimination** consists of the following steps :

(1) *We multiply both the equations by such numbers so as to make the co-efficients of one of the two unknowns numerically the same.*

(2) *Then we add or subtract so as to get an equation containing only the other unknown. By solving this equation, we get the value of the one unknown.*

(3) *We substitute the value of this unknown in either of the equations.*

By solving that, we get the value of the other unknown.

EXERCISE 1 (c)

(Section A)

Solving the following systems of equations :

1. $x+y=10$
   $2x-3y=5.$
2. $x+y-5=0$
   $-2x+y=2.$
3. $3x-4y=5$
   $5x+2y=17.$
4. $4x-3y=4$
   $2x+5y=15.$
5. $x-2y=5$
   $3x-4y=13.$
6. $2x-3y=11$
   $3x-y=6.$

(Section B)

Solve each of the following systems of equations :

7. $3x+y=4$
   $y-4x=3$
8. $x-2y=2$
   $5x+5y=4$
9. $x+y=10$
   $2x-y=1$
10. $x-2y=13$
    $y=7x+6.$

Example 5. Solve

$$x+7y=21$$
$$5x-17y=1$$

Solution.

$$x+7y=21 \quad \ldots(1)$$
$$5x-17y=1 \quad \ldots(2)$$

We transform the first equation to obtain x in terms of y.

From equation (1), we get $\quad x=21-7y.$

We substitute the expression for x in the equation (2).

$$5(21-7y)-17y=1$$

or $\quad 105-35y-17y=1$

or $\quad -52y=1-105$

or $\quad -52y=-104$

$\therefore \quad y=2.$

Then we substitute this value of y in equation (1) to obtain the value of x.

$$x+7\times2=21$$

or $\quad x+14=21$

or $\quad x=7$

$\therefore$

The solution is $x=7, y=2.$

This method of **substitution** consists of the following steps :

(1) *From either of the given equations, we express one of the two unknowns in terms of the other.*

(2) *We substitute the value of unknown thus expressed in the other equation. By solving this equation, we get the value of one unknown.*

(3) *We substitute the value of this unknown in either of the equations. By solving that, we get the value of the other unknown.*

EXERCISE 1 (d)

(Section A)

Solve the following systems of equations :

1. $2x-3y=7$, $5x+y=9$.

2. $2x+3y=8$, $4x=4+6y$.

3. $15x-8y=29$, $17x+12y=75$.

4. $8x+13y-29=0$, $12x-7y-17=0$.

Solve each of the following systems of equations :

5. $2x+7y=39$, $3x+5y=31$.

6. $12x+15y=-18$, $18x-7y=-86$.

7. $2x+y-3=0$, $y-3x-1=0$

8. $y=4x-7$, $16x-5y=25$

9. $6x=7y+7$, $7y-x=8$

10. $3x-4y=20$, $x+2y=5$.

Example 6. Solve

$$3x+y=17$$
$$8x+11y=37$$

Solution.

$$3x+y=17 \quad \ldots(1)$$
$$8x+11y=37 \quad \ldots(2)$$

From equation (1), we have $\quad y=17-3x$

From equation (2), we have $\quad 11y=37-8x$

$$\Rightarrow \quad y=\frac{37-8x}{11}$$

Equating these values of y, we get

$$17-3x=\frac{37-8x}{11}$$

or $\quad 187-33x=37-8x$

or $\quad -33x+8x=37-187$

or $\quad -25x=-150$

$\therefore \quad x=6$

Substituting the value of x in equation (1), we have

$$3\times 6+y=17 \quad \text{or} \quad 18+y=17$$
$$y=17-18 \qquad y=-1$$

Hence the solution is $x=6$, $y=-1$.

This method of **comparison** consists of the following steps :

(1) *Express the same variable in terms of the other in both the equations.*

(2) *Equate the results and obtain a simple equation containing only one variable.*

(3) *Solve it and find the value of the variable.*

(4) *Substitute the value of this variable in one of the equations and find the value of the other variable.*

EXERCISE 1 (e)

(Section A)

Solve the following systems of equations :

1. $x-y=0$
$2x-y=-1$

2. $3x-y=2$
$x+2y=3$

3. $3x+y=18$
$3x-4y=3$

4. $5x+y=11$
$x+5y=7$

(Section B)

Solve each of the following systems of equations :

5. $7x+4y=5$
$5x+6y=2$

6. $x-y+1=0$
$2x+2y+3=0$

7. $5x-15y=22$
$7x+10y=37$

8. $11x+15y+23=0$
$7x-2y=20$

9. $6x-5y=21$
$5x+4y=17\frac{1}{2}$

10. $5x+4y+8{\cdot}7=0$
$3x+y+4{\cdot}1=0$

Example 7. Solve the following equations :

$$\frac{x}{2}+y=0{\cdot}8, \qquad \frac{7}{x+\frac{y}{2}}=10.$$

Solution. The given equations are [*C.B.S.E., 1984 (A.I.)*]

$$\frac{x}{2}+y=0{\cdot}8 \quad \text{or} \quad x+2y=1{\cdot}6 \qquad ...(1)$$

$$\frac{7}{x+\frac{y}{2}}=10 \quad \text{or} \quad 7=10x+5y$$

$$\text{or} \quad 10x+5y=7 \qquad ...(2)$$

Multiplying equation (1) by 10, we get

$$10x+20y=16 \qquad ...(3)$$

Subtracting (2) from (3), we get

$$15y=9 \qquad \therefore \quad y=\frac{3}{5}$$

Substituting the value of y in equation (1), we get

$$x+2\times\frac{3}{5}=\frac{8}{5} \quad \text{or} \quad x=\frac{8}{5}-\frac{6}{5}$$

$$\therefore \quad x=\frac{2}{5}$$

Hence $\quad x=\frac{2}{5}, \qquad y=\frac{3}{5}.$

Example 8. Solve : $\frac{5}{y}-\frac{2}{x}=1\frac{1}{6}$

$$\frac{36}{x}-\frac{24}{y}=1$$

Solution. The given equations are

$$\frac{-2}{x}+\frac{5}{y}=\frac{7}{6} \quad \text{..(1)}$$

$$\frac{36}{x}-\frac{24}{y}=1 \quad \text{...(2)}$$

Multiplying both sides of equation (1) by 18, we get

$$\frac{-36}{x}+\frac{90}{y}=21 \quad \text{...(3)}$$

Adding equations (2) and (3), we get

$$\frac{66}{y}=22$$

or $\frac{1}{y}=\frac{22}{66}$ or $\frac{1}{y}=\frac{1}{3}$

$\therefore$ $y=3$

Substituting the value of y in equation (1), we get

$$\frac{-2}{x}+\frac{5}{3}=\frac{7}{6}$$

or $\frac{-2}{x}=\frac{7}{6}-\frac{5}{3}$ or $\frac{-2}{x}=-\frac{3}{6}$

or $\frac{-2}{x}=\frac{-1}{2}$ $\therefore$ $x=4$

Hence $x=4$, $y=3$

Example 9. Solve : $10x+3y=30xy$

$$5x-12y=7xy$$

Solution. The given equations are

$$10x+3y=30xy$$

$$5x-12y=7xy$$

These equations are *not* linear in x and y, but we can convert them into linear equations. However, we will put them in the form of equations of Example 8, worked out above.

Dividing both equations by x and y, we get

$$\frac{10}{y}+\frac{3}{x}=30$$

$$\frac{5}{y}-\frac{12}{x}=7$$

Solve these equations, as solved in worked out Example 8.

Observe that equations (1) and (2) are not linear in x and y. But these can be converted into linear equation by an appropriate substitution *e.g.* by putting u for $\frac{1}{x}$ and v for $\frac{1}{y}$. Then equations become

$$3u+10v=30, \qquad -12u+5v=7.$$

EXERCISE 1 (f)

(Section A)

Solve the following equations algebraically :

1. $2x+3y=6$

$x-\frac{5}{4}y=1.$

2. $5x+4y=8{\cdot}7$

$3x+y=4{\cdot}1.$

3. $3a-2b=8$

$\frac{a}{3}+\frac{b}{2}=\frac{5}{4}$

4. $5x+2y=11$

$\frac{x}{4}-\frac{y}{3}=1\frac{5}{12}$

(Section B)

5 $\frac{x}{2}+\frac{y}{4}=6$

$\frac{x}{5}-\frac{y}{2}=0.$

6. $4x+\frac{6}{y}=15$

$6x-\frac{8}{y}=14.$

7. $\frac{3}{x}-4y=10$

$\frac{4}{x}+3y=5.$ [*C.B.S.E.*, *1984* (*Delhi*)]

8. $\frac{4}{x}+5y=7.$

$\frac{3}{x}+4y=5.$ [*C.B.S.E.*, *1983* (*Delhi*)]

(Section C)

9. $\frac{1}{7x}+\frac{1}{6y}=3$

$\frac{1}{2x}-\frac{1}{3y}=5.$

10. $x-y=0{\cdot}9$

$\frac{11}{2(x+y)}=1.$

11. $\frac{x-1}{y+1}=\frac{3}{4}$

$\frac{x+2}{y-2}=\frac{4}{3}$

12. $\frac{7+x}{5}-\frac{2x-y}{4}=3y-5$

$\frac{5y-7}{2}+\frac{4x-3}{6}=18-5x.$

13. $5x+4y=31xy$

$3y-2x=6xy.$

14. $\frac{x+y}{xy}=2$

$\frac{x-y}{xy}=6.$

1·4. GENERAL SOLUTION AND CONDITIONS FOR SOLVABILITY

Let us solve the general system of two simultaneous linear equations given below :

$$a_1x+b_1y+c_1=0 \quad \text{...(1)}$$

$$a_2x+b_2y+c_2=0 \quad \text{...(2)}$$

We eliminate y by using one of the methods discussed in the last Article 1·3.

Multiplying both sides of (1) by b_2 and of (2) by b_1, we get

$$a_1b_2x+b_1b_2y+b_2c_1=0 \quad \text{...(3)}$$

$$a_2b_1x+b_1b_2y+b_1c_2=0 \quad \text{...(4)}$$

Subtracting (4) from (3), we get

$$(a_1b_2-a_2b_1)x+b_2c_1-b_1c_2=0$$

Then $\quad (a_1b_2-a_2b_1)x=b_1c_2-b_2c_1 \qquad ...(5)$

Multiplying both sides of (1) by a_2 and of (2) by a_1, we get

$$a_1a_2x+a_2b_1y+a_2c_1=0 \qquad ...(6)$$

$$a_1a_2x+a_1b_2y+a_1c_2=0 \qquad ...(7)$$

Subtracting (7) from (6), we get

$$(a_2b_1-a_1b_2)y+a_2c_1-a_1c_2=0$$

Then $\quad (a_2b_1-a_1b_2)y=a_1c_2-a_2c_1$

or $\quad (a_1b_2-a_2b_1)y=c_1a_2-c_2a_1 \qquad ...(8)$

Now two cases arise : Either $a_1b_2-a_2b_1\neq0$ or $a_1b_2-a_2b_1=0$

(*i*) If $a_1b_2-a_2b_1\neq0$, then we get from (5) and (8),

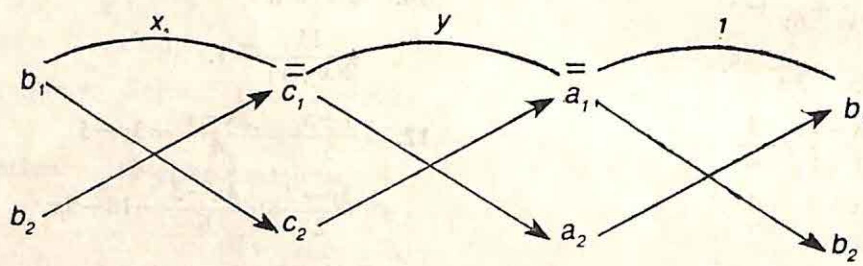
$$x=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \qquad y=\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}.$$

This forms the one and only solution of the given system. So, we conclude that if $a_1b_2-a_2b_1\neq0$ *i.e.*, $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$, the system has a *unique solution* which is generally written as

$$\frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}$$

This is known as **method of cross multiplication.**

The above solution can be written easily with the help of the following diagram :



Here the co-efficients are to be multiplied crosswise in the direction of arrows. Products formed by descending from left to right are positive, but those formed by descending from right to left are negative.

If $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$, the system of equations has a *unique solution.* Hence, the system of equations is **consistent**, when $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$.

(*ii*) If $a_1b_2-a_2b_1=0$, then $\frac{a_1}{a_2}=\frac{b_1}{b_2}$. Let us suppose that $\frac{a_1}{a_2}=\frac{b_1}{b_2}=k.$

Thus $\quad a_1=ka_2$ and $b_1=kb_2$.

So, the given equations become

$$ka_2x+kb_2y+c_1=0$$

$$a_2x+\ b_2y+c_2=0$$

These equations can both be satisfied simultaneously only if $c_1=kc_2$. If $c_1=kc_2$, any solution of $a_2x+b_2y+c_2=0$ will satisfy $a_1x+b_1y+c_1=0$ and any solution of $a_1x+b_1y+c_1=0$ will satisfy $a_2x+b_2y+c_2=0$. It follows that there are *infinitely many solutions* of the system, if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=k$ and $c_1=kc_2$. Thus, if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=k$ and $\frac{c_1}{c_2}=k$ *i.e.*, $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$, the system of equations is **dependent**.

If $\frac{a_1}{a_2}=\frac{b_1}{b_2}$ and $c_1\neq kc_2$, there is *no solutions* of the system. Thus, if $\frac{a_1}{a_2}=\frac{b_1}{b_2}$ and $\frac{c_1}{c_2}\neq k$ *i.e.*, $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$, the system of equations is **inconsistent**.

Example 10. Find out if the following systems of equations are consistent, inconsistent or dependent.

(*a*) $3x=y+8$
$9x-3y=12$

(*b*) $2x+y=4$
$3x+y+3=0$

(*c*) $x-2y=1$
$3x-3=6y$

Solution. First we write the equations of the system in the general form.

(*a*) Given equations are $\quad 3x=y+8$
$$9x-3y=12$$

These equations can be written as

$$3x-y-8=0$$
$$9x-3y-12=0$$

Here $\quad \frac{a_1}{a_2}=\frac{3}{9}$ *i.e.*, $\frac{1}{3}$, $\frac{b_1}{b_2}=\frac{-1}{-3}$ *i.e.*, $\frac{1}{3}$, $\frac{c_1}{c_2}=\frac{-8}{-12}$ *i.e.*, $\frac{2}{3}$

$$\therefore \quad \frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}.$$

Hence the system of equations is inconsistent.

(*b*) Given equations are $\quad 2x+y=4$
$$3x+y+3=0$$

These equations can be written as

$$2x+y-4=0$$
$$3x+y+3=0$$

Here $\quad \frac{a_1}{a_2}=\frac{2}{3}$, $\frac{b_1}{b_2}=1$ and $\frac{c_1}{c_2}=\frac{-4}{3}$

$$\therefore \quad \frac{a_1}{a_2}\neq\frac{b_1}{b_2}.$$

Hence the system of equations is consistent.

(*c*) Given equations are $\quad x-2y=1$
$$3x-3=6y$$

These equations can be written as

$$x-2y-1=0$$
$$3x-6y-3=0$$

Here $\quad \frac{a_1}{a_2}=\frac{1}{3}$, $\frac{b_1}{b_2}=\frac{-2}{-6}$ *i.e.*, $\frac{1}{3}$ and $\frac{c_1}{c_2}=\frac{-1}{-3}$ *i.e.*, $\frac{1}{3}$

$$\therefore \quad \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$$

Hence the system of equations is dependent.

Example 11. Find the value of k for which the system

$$3x+ky=8$$
$$2x-y+5=0$$

will have (*a*) a unique solution (*b*) no solutions.

Solution. The given equations are

$$3x+ky-8=0$$
$$2x-y+5=0$$

Here $\quad \frac{a_1}{a_2}=\frac{3}{2},\ \frac{b_1}{b_2}=\frac{k}{-1}$ and $\frac{c_1}{c_2}=\frac{-8}{5}$

(*a*) The system has a unique solution, if $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$

i.e., $\quad \frac{3}{2}\neq\frac{k}{-1}$ or $k\neq-\frac{3}{2}$

(*b*) The system has no solution, if $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$

i.e., $\quad \frac{3}{2}=\frac{k}{-1}$ or $k=-\frac{3}{2}$.

EXERCISE 1 (g)

(Section A)

1. Determine whether the following systems of equations are consistent, inconsistent or dependent :

(*a*) $3x-y=2$, $6x-2y=3$

(*b*) $4y=3x+23$, $4x=11-3y$

(*c*) $2x+3y-6=0$, $6x+9y-18=0$

(*d*) $4x-y=7$, $12x-21=3y$

2. In each of the following systems of equations determine whether the system has a unique solution, no solutions or infinitely many solutions :

(*a*) $x+y=3$, $2x+3y=8$

(*b*) $x+y=5$, $3x+3y=15$

(*c*) $x-2y+5=0$, $3x-6y+12=0$

(*d*) $3x=2+y$, $2y=3-x$

(Section B)

3. Find the values of k for which the system of equations

$$2x+ky=1$$
$$3x-5y=7$$

has (*a*) a unique solution, (*b*) no solutions.

4. Find the values of k for which the system of equations

$$kx+2y-5=0$$
$$y+3x=1$$

will have (*a*) a unique solution and (*b*) no solutions.

Example 12. Solve the following system of equations by the method of cross-multiplication :

$$8x=7y+19$$
$$10x=23+9y$$

Solution. The given equations are

$$8x=7y+19$$
$$10x=23+9y$$

Writing these equations in general form, we get

$$8x+(-7)y+(-19)=0$$
$$10x+(-9)y+(-23)=0$$

By the method of cross-multiplication, we have

$$\frac{x}{(-7)\times(-23)-(-9)\times(-19)}=\frac{y}{(-19)\times10-(-23)\times8}=\frac{1}{8\times(-9)-10\times(-7)}$$

or $$\frac{x}{161-171}=\frac{y}{-190+184}=\frac{1}{-72+70}$$

or $$\frac{x}{-10}=\frac{y}{-6}=\frac{1}{-2} \text{ i.e., } \frac{x}{10}=\frac{y}{6}=\frac{1}{2}$$

$\therefore$ $$x=\frac{10}{2} \text{ i.e., } 5 \text{ and } y=\frac{6}{2} \text{ i.e., } 3.$$

Note that the method of cross-multiplication can be used only when the given system is *consistent i.e.*, has a unique solution. In general, first check that the system is consistent and then use the method of cross multiplication.

Example 13. Solve the following system of equations :

$$(a+c)x-(a-c)y=2ab$$
$$(a+b)x-(a-b)y=2ab$$

Solution. The given equations can be written as

$$(a+c)x-(a-c)y-2ab=0$$
$$(a+b)x-(a-b)y-2ab=0$$

Here $$\frac{a_1}{a_2}=\frac{a+c}{a+b} \text{ and } \frac{b_1}{b_2}=\frac{-(a-c)}{-(a-b)} \text{ i.e., } \frac{a-c}{a-b}$$

So, $$\frac{a_1}{a_2}\neq\frac{b_1}{b_2}.$$

Hence the given system is consistent.

By the method of cross-multiplication, we get

$$\frac{x}{-(a-c)\times(-2ab)-[-(a-b)]\times(-2ab)}=\frac{y}{(-2ab)(a+b)-(-2ab)(a+c)}$$
$$=\frac{1}{(a+c)\times[-(a-b)]-(a+b)\times[-(a-c)]}$$

or $$\frac{x}{-2ab(-a+c+a-b)}=\frac{y}{-2ab(a+b-a-c)}$$
$$=\frac{1}{-a^2+ab-ca+bc+a^2-ca+ab-bc}$$

or $$\frac{x}{-2ab(c-b)}=\frac{y}{-2ab(b-c)}=\frac{1}{2ab-2ca}$$

or $$\frac{x}{2ab(b-c)}=\frac{y}{-2ab(b-c)}=\frac{1}{2a(b-c)}$$

$\therefore$ $$x=\frac{2ab(b-c)}{2a(b-c)},\quad y=\frac{-2ab(b-c)}{2a(b-c)}$$

i.e., $x=b,\quad y=-b$

EXERCISE 1 (h)

(Section A)

Solve by the method of cross-multiplication :

1. $x+4y=14$
   $7x-3y=5$

2. $x=2y+6$
   $y=2x-3$

3. $13x+7y+114=0$
   $26x-10y+60=0$

4. $3x=4y+20$
   $x+2y=5$

(Section B)

Solve the following systems of equations :

5. $\frac{x}{3}+\frac{y}{2}=3$
   $x-2y=2$

6. $\frac{x}{4}+\frac{y}{3}=6$
   $\frac{x}{3}+\frac{y}{4}=6\frac{1}{4}$

7. $ax+by=a^2+b^2$
   $ax-by=a^2-b^2$

8. $ax+by=2ab$
   $x+y=a+b$

9. $x=py+q$
   $y=qx+p$

10. $px+qy=r$
    $qx=py$

(Section C)

Solve the following systems of equations :

11. $(a+b)x-(a-b)y=3ab$
    $(a+b)y-(a-b)x=ab$

12. $(a+b)x+(a-b)y=2a$
    $(a-b)x+(a+b)y=2b$

1·5. WORD PROBLEMS

Many everyday problems can be easily solved by *translating* them into a system of equations. A few types of problems are given here.

The method of problem-solving consists of three steps :

(1) *Translating the word problem into symbolic language.*

(2) *Solving the equations, and*

(3) *Interpreting the solution of the equations.*

Example 14. In a two-digit number, the sum of the digits is 7. The number formed by reversing the digits is 9 more than the original number. Find the number.

Solution. Let x represent the digit in the units place and y represent the digit in the tens place.

Then $$y+x=7$$

The number is $10y+x$.

When we interchange the digits, x becomes the digit in the tens place and y the digit in the units place.

The new number $=10x+y$.

The problem states that

$$10x+y=10y+x+9$$

or $$10x+y-10y-x=9$$

or $$9x-9y=9$$

or $$x-y=1$$

We thus have the system of equations :

$$x+y=7$$
$$x-y=1$$

Adding, we get $2x=8$ or $x=4$

Substituting $x=4$ in $x+y=7$, we get

$$4+y=7 \quad \text{or} \quad y=3.$$

Therefore, the required number$=10\times3+4$
$=34.$

Example 15. A fraction becomes 1 when 8 is added to its numerator. It becomes $\frac{1}{5}$ when 1 is subtracted from its numerator and its denominator is multiplied by 2. Find the fraction.

Solution. Let the fraction be $\frac{x}{y}$.

When 8 is added to the numerator, the fraction becomes 1.

Then $$\frac{x+8}{y}=1$$

or $$x+8=y$$

or $$x-y=-8$$

When 1 is subtracted from the numerator and the denominator is multiplied by 2, the fraction becomes $\frac{1}{5}$.

Then $$\frac{x-1}{2y}=\frac{1}{5}$$

or $$5x-5=2y$$

or $$5x-2y=5$$

We thus have the system of equations :

$$x-y=-8 \quad \text{...(1)}$$
$$5x-2y=5 \quad \text{...(2)}$$

Solving equations (1) and (2), we get

$$x=7, \quad y=15.$$

Therefore, the required fraction$=\frac{7}{15}$.

Example 16. Three years hence a father will be three times as old as his son, 7 years ago he was seven times as old as the son. How old are they now ?

Solution. Let the present age of the father be x years and of the son be y years.

After 3 years, the father's age will be $(x+3)$ years.

After 3 years, the son's age will be $(y+3)$ years.

Thus $\quad x+3=3(y+3)$

or $\quad x+3=3y+9$

or $\quad x-3y=6$

7 years ago, the father's age was $(x-7)$ years.

7 years ago, the son's age was $(y-7)$ years.

Thus $\quad x-7=7(y-7)$

or $\quad x-7=7y-49$

or $\quad x-7y=-42$

We thus have the following system of equations :

$x-3y=6$ and $x-7y=-42$

Solving these equations, we get

$x=42, \qquad y=12$

Father's age=42 years, $\qquad$ Son's age=12 years.

EXERCISE 1 (i)

Number Problems

1. The sum of two numbers is 45 and their difference is 15. Find the numbers.
2. Find two numbers, which differ by 7, such that twice the greater added to five times the smaller makes 42.
3. Find two numbers such that twice the first added to the second makes 21, and twice the second added to the first makes 27.
4. Find two numbers such that four times the first added to three times the second is 93 and the excess of three times the first over twice the second is 6.
5. If I add 1 to each of the two given numbers, their ratio is 1 : 2. If I subtract 5 from each the ratio is 5 : 11. Find the numbers.

Digit Problems

6. A number of two digits is four times the sum of its digits. If 9 be added to the number, the digits in the number are reversed. Find the number.
7. A number of two digits is four times the sum of its digits and the number formed by reversing the digits is 27 more than the original number. Find the number.
8. The sum of the digits of a two-digit number is 7. If the digits are reversed, the new number increased by 3 equals four times the original number. Find the original number.
9. The units digit of a two digit number is twice the difference of its digits. If the order of the digits is reversed the number is increased by 18. Find the number.
10. In a two-digit number, the sum of the digits is 13. If the number is subtracted from the one obtained by interchanging the digits, the result is 45. Find the original number. [*C.B.S.E., 1978 (Delhi)*]
11. A number consists of two digits, and the tens digit is 2/3 of the units digit. If the digits are reversed the number is increased by 27. Find the number.
12. The result of dividing a number of two digits by the number with the digits reversed is $1\frac{3}{4}$. If the sum of the digits is 12, find the number.

Fraction Problems

13. When the numerator of a fraction is increased by 4, the fraction increases by $\frac{2}{3}$. What is the denominator of the fraction ? [*C.B.S E., 1978 (A.I.)*]
14. A fraction becomes 2 when 9 is added to its numerator and it becomes 1 when 2 is subtracted from its denominator. Find the fraction.

15. Find the fraction which becomes $\frac{1}{2}$ when the denominator is increased by 4 and equals $\frac{1}{3}$ when the numerator is diminished by 5.

16. If 1 is added to the denominator of a fraction, the fraction becomes $\frac{1}{2}$. If 1 is added to the numerator of the fraction, the fraction becomes 1. Find the fraction.

17. A fraction becomes equal to $\frac{1}{3}$ when 1 is subtracted from its numerator and it becomes equal to $\frac{1}{4}$ when 8 is added to its denominator. Find the fraction.

18. Find the fraction which becomes $\frac{2}{3}$ when numerator and denominator are increased by 1 and $\frac{1}{2}$ when numerator and denominator are diminished by 1.

19. If the numerator of a fraction is increased by 2 and the denominator by 1, it becomes equal to $\frac{5}{8}$, and if the numerator and the denominator of the same fraction are each increased by 1, the fraction becomes equal to $\frac{1}{2}$. Find the fraction.

Age Problems

20. Six years ago Jai Ram was three times as old as his son. In six years time, he will be twice as old as his son. Find their present ages.

21. Five years hence the age of a man will be 3 times that of his son. Five years ago, the father's age was 7 times that of his son. What are their ages now ?

22. Ten years ago, father was twelve times as old as his son and ten years hence he will be twice as old as his son will be. Find their present ages. [*C.B.S.E., 1981 (A.I.)*]

23. Eight years ago Ravi's age was $\frac{3}{4}$ of Pramod's. Four years hence Ravi's age will be $\frac{6}{7}$ of Pramod's. Find their present ages.

24. Three times my father's age added to seven times my sister's age is 183 years. Six times the difference between their ages added to 9 is three times the sum of their ages. Find their ages.

Miscellaneous Problems

25. Mohan Lal sold 9 bags and 6 pens for Rs. 90. Again he sold 8 bags and 5 pens at the same rate for Rs. 77. Find the price of one bag and one pen.

26. 3 nuts and 6 bolts weigh 72 g, 4 nuts and 5 bolts weigh 66 g. Find the weight of a nut and a bolt.

27. Prachi has 50 coins, some are 50 P coins and the rest 10 P coins. The total value of all the coins is Rs. 7·80. Find the number of each kind of coin.

28. In an examination, Manju's marks are 2 more than $\frac{2}{5}$ of Kavita's. If Manju scored 4 less she would have had $\frac{3}{8}$ of Kavita's. Find their marks.

29. A man buys postage stamps of denominations 3 paise and 5 paise for Re 1. He buys 22 stamps in all. Find the number of 3 paise stamps bought by him.

30. There were 2,500 persons who bought tickets to see a village fair. The adults paid 75 paise each for their admission tickets but the children paid only 25 paise each. If the total receipts amounted to Rs. 1,503, using an equation method, find how many adults and how many children saw the fair ?

31. A boy saves his pocket money to buy a toy aeroplane. He finds that the cost of the toy is Rs. 50 less than his yearly pocket money. What is his monthly pocket money if he could buy the toy out of 3 months pocket money and Rs. 40 extra which he gets from his mother ?

REVIEW EXERCISE 1

(Section A)

1. Fill in the blanks to make the following statements true :

(a) An equation of the type $ax+by+c=0$, where a, b and c are real numbers and a, b are not simultaneously zero is called a.........equation. [*C.B.S.E., 1980 (A.I.)*]

(b) The graph of the equation $x+y=10$ is a.........

(c) The graph of $x=1$ is a line parallel to.........axis. [*C.B.S.E., 1978 (Delhi)*]

(d) The graph of $y=1$ is a line parallel to.........axis. [*C.B.S.E., 1980 (A.I.)*]

(e) A system of linear equations is inconsistent if it has.........solutions.

(f) The method of cross-multiplication can be used only when the given system of linear equations is..........

2. If $(5, k)$ is a solution of the equation $2x+y-7=0$. Find the value of k.

3. Solve : $y=2x-6$, $y=0$.

(Section B)

4. Solve the following system of equations :

$$11x-8y=27, \qquad 3x+5y=-7$$

[*C.B.S.E., 1982 (Delhi)*]

5. Solve for x and y :

$$9x+4y=29$$
$$x-y=9.$$

6. Solve the simultaneous equations, algebraically

$$2x+y=6$$
$$3y=8+4x$$

7. Solve : $\dfrac{x}{4}-3=\dfrac{y}{6}$, $\qquad \dfrac{1}{2}x-y=-2.$

8. Solve graphically the simultaneous equations :

$$x+y=7, \qquad 2x-3y=9.$$

[*C.B.S.E., 1978 (A.I.)*]

9. Sketch the graphs of the equations :

$$2x+3y=6 \quad \text{and} \quad 6x-5y=4$$

Indicate in the graph the solution set of the above equations. [*C.B.S.E., 1979 (Delhi)*]

10. Solve : $\dfrac{7}{x}+\dfrac{8}{y}=2$, $\qquad \dfrac{2}{x}+\dfrac{12}{y}=20.$

(Section C)

11. A lady has only 10 paise and 25 paise coins in her purse. If in all she has 60 coins totalling Rs. 8·25, how many of each does she have ? [*C.B.S.E., 1980 (A.I.)*]

12. The total cost of 8 buckets and 5 mugs is Rs. 92 and the total cost of 5 buckets and 8 mugs is Rs. 77. Find the cost of 2 mugs and 3 buckets.

13. The sum of the digits of a two-digit number is 9. If the digits are reversed, the number obtained is 45 more than the original number. Find the original number. [*C.B.S.E., 1984 (A.I,)*]

14. The sum of the numerator and denominator of a fraction equals 7. Four times the numerator is 8 less than 5 times the denominator. What is the fraction ?

15. In a shooting competition a markman receives 50 paise if he hits the mark and pays 20 paise if he misses it. He tried 60 shots and was paid Rs. 1·30. How many times did he hit the mark ?

16. Solve the following system of equations :

$$\frac{1}{7x}+\frac{1}{6y}=3, \qquad \frac{1}{2x}-\frac{1}{3y}=5$$

17. The present age of a father is 3 years more than three times the age of the son. Three years hence, father's age will be 10 years more than twice the age of the son. Find their present ages. *[C.B.S.E., 1982 (Delhi)]*

18. There are two examination rooms A and B. If 10 candidates are sent from A to B, the number of students in each room is the same. If 20 students are sent from B to A, the number of students in A is double the number of students in each room. *[C.B.S.E., 1984 (A.I.)]*

19. A and B, each has a certain number of mangoes. A says to B, "if you give me 30 of your mangoes, I will have twice as many as left with you." B replies, "if you give me 10, I will have thrice as many as left with you." How many mangoes does each have ? *[C.B.S.E., 1983 (A.I.)]*

20. A sailor goes 8 km downstream in 40 minutes and returns in 1 hour. Determine the speed of the sailor in still water and tbe speed of the current. *[C.B.S.E., 1977 (Delhi)]*

□□

2

QUADRATIC EQUATIONS

2·1. QUADRATIC POLYNOMIALS

You have already studied **quadratic polynomials** in one variable.

Consider the following quadratic polynomials :

(*i*) x^2-4 (*ii*) $25x^2-9$

(*iii*) x^2-5x+6 (*iv*) $6x^2-x-2$.

Note that all the co-efficients in each of the above polynomials are real numbers. The *general form* of a quadratic polynomial is $\mathbf{ax^2+bx+c}$, where a, b, c are real numbers, $a \neq 0$ and x is a variable. Throughout this chapter we will consider x to be a real variable. So, in a quadratic polynomial x can take any real value.

2·2. ZEROS OF A QUADRATIC POLYNOMIAL

Every quadratic polynomial in x has a real value for every real value of x.

Let us consider the quadratic polynomial $p(x)=3x^2+2x-1$. If we substitute a real value of x, say $x=0$, the value of $p(x)$ is -1. If we take $x=1$, the value of $p(x)$ is 4 and so on.

Can you find those values of x for which $p(x)=0$?

If we substitute $x=-1$, the value of $p(x)$ is $3-2-1$ *i.e.*, zero.

If we substitute $x=\frac{1}{3}$, the value of $p(x)$ is $3\times\frac{1}{9}+2\times\frac{1}{3}-1$ or $\frac{1}{3}+\frac{2}{3}-1$ *i.e.*, zero.

Thus, there are two values of x *i.e.*, $x=-1$ and $x=\frac{1}{3}$ for which the value of polynomial becomes *zero*. These two values of x are called the **zeros** of the quadratic polynomial $3x^2+2x-1$.

If k is a real number and the value of a quadratic polynomial $\mathbf{ax^2+bx+c}$ becomes zero for $\mathbf{x=k}$, then the real number k is called a zero of the quadratic polynomial $\mathbf{ax^2+bx+c}$.

Every quadratic polynomial can have *at most two zeros*. This fact cannot be proved here, since the proof is beyond the scope of the book.

Consider the quadratic polynomial x^2+2. There is no real value of x which makes the polynomial zero. For every real value of x, $x^2 \geqslant 0$ which means $x^2+2 \geqslant 2$. Thus, *some quadratic polynomials do not have any real zero*.

If we are given a quadratic polynomial, we can test whether any given real number k is a zero of the given polynomial or not by substitution.

If we are given a quadratic polynomial, how do we determine its zeros ?

Last year you solved linear equations in one variable *i.e.*, $ax+b=0$, $a\neq 0$. This really amounts to finding zeros of the linear polynomial $ax+b$. Similarly, for finding zeros of a quadratic polynomial ax^2+bx+c, $a\neq 0$, we have to solve $ax^2+bx+c=0$ which is called **quadratic equation.**

The following are all **quadratic equations** :

$$x^2-4=0 \qquad 25x^2-9=0$$

$$x^2-5x+6=0 \qquad 6x^2-x-2=0$$

A quadratic equation in one variable is an equation in one variable which equates to zero a polynomial of degree two.

Every quadratic equation can, therefore, be written in the form

$$\mathbf{ax^2+bx+c=0}, \qquad \mathbf{a\neq 0}$$

where the coefficients a, b, c may belong to any of the number systems at our disposal.

Thus the *general quadratic equation* is

$\mathbf{ax^2+bx+c=0}$, where a, b, c, $\in$ R and $a\neq 0$.

If the real numbers α and β are two zeros of a quadratic polynomial ax^2+bx+c, $a\neq 0$, we say that α and β are the two **roots** of the quadratic equation $ax^2+bx+c=0$, $a\neq 0$.

Example 1. Show that $\frac{1}{2}$ is a zero of the polynomial $2x^2+7x-4$.

Solution. Let $p(x)=2x^2+7x-4$

The value of $p(x)$ for $x=\dfrac{1}{2}$ is $2\left(\dfrac{1}{2}\right)^2+7\times\dfrac{1}{2}-4$

$$=2\times\frac{1}{4}+\frac{7}{2}-4$$

$$=\frac{1}{2}+\frac{7}{2}-4$$

$$=4-4 \qquad =0$$

Thus, the value of $p(x)$ for $x=\frac{1}{2}$ is zero.

Hence $\frac{1}{2}$ is a zero of the polynomial $2x^2+7x-4$.

Example 2. For the quadratic equation $2x^2-5x-3=0$, determine which of the following are solutions ?

(*a*) $x=3$ (*b*) $x=-1$ (*c*) $x=-\frac{1}{2}$.

Solution. The given equation is $2x^2-5x-3=0$.

(*a*) Substituting $x=3$ in the L.H.S. of the equation we get

$$2\times 3^2-5\times 3-3=18-15-3=0$$

$\therefore$ L.H.S.=R.H.S. for $x=3$.

Hence $x=3$ is a solution of the given quadratic equation.

(*b*) Substituting $x=-1$ in the L.H.S. of the equation, we get

$$2\times(-1)^2-5(-1)-3=2+5-3=4$$

$\therefore$ L.H.S.$\neq$0 or L.H.S.$\neq$R.H.S.

Hence $x=-1$ is not a solution of the given equation.

(*c*) Substituting $x=-\frac{1}{2}$ in the L.H.S. of the equation, we get

$$2\times\left(-\frac{1}{2}\right)^2-5\left(-\frac{1}{2}\right)-3=2\times\frac{1}{4}+\frac{5}{2}-3$$

$$=\frac{1}{2}+\frac{5}{2}-3 \qquad =0$$

$\therefore$ L.H.S.=R.H.S. for $x=-\frac{1}{2}$.

Hence $x=-\frac{1}{2}$ is a solution of the given equation.

EXERCISE 2 (a)

(Section A)

1. Which of the following are quadratic equations ?
   (a) $x^2-5x+6=0$ (b) $2x^2+7x=0$
   (c) $2x^2+3x=2$ (d) $6x^2+1=5x$
   (e) $x^3+x^2-x+1=0$ (f) $x^3+4=2x^3$
   (g) $3x^2-4x+2=2x^2-2x+4$.
2. Show that -2 is a zero of the polynomial $3x^2+11x+10$.
3. Show that 2 is not a zero of the polynomial $x^2-7x+14$.

(Section B)

4. Which of the numbers 2, 3 and -4 are the zeros of the polynomial $2x^2+7x-4$?
5. Which of the numbers $\frac{2}{3}$, 1 and $\frac{3}{2}$ are the zeros of the polynomial $6x^2-13x+6$?
6. For the quadratic equation $2x^2+3x-2=0$, determine which of the following are solutions ?
   (a) $x=-2$ (b) $x=0$ (c) $x=\frac{1}{2}$.
7. For the quadratic equation $6x^2-x-15=0$, determine which of the following are solutions ?
   (a) $x=\frac{3}{5}$ (b) $x=\frac{5}{3}$ (c) $x=-\frac{3}{2}$.

2·3. SOLVING A QUADRATIC EQUATION BY FACTORIZATION

If a and b are any two real numbers such that ab=0, then either a=0 or b=0 or both a=0 and b=0.

Given a quadratic equation, if we can factorise the quadratic polynomial into two *linear factors*, we can set each factor equal to zero and obtain a solution.

Example 3. Solve $x^2=49$, $x \in R$.

Solution.

$$x^2=49$$

or $$x^2-49=0$$ Factorise the quadratic polynomial

or $$(x+7)(x-7)=0$$ x^2-49. $[x^2-a^2=(x+a)(x-a)]$

$\therefore$ $$x+7=0 \quad \text{or} \quad x-7=0$$

then $$x=-7 \quad \text{or} \quad x=7$$

Thus, the two solutions of the given quadratic equation are -7 and 7.

Example 4. Solve $2x^2-5x=0$, when $x \in R$.

Solution.

$$2x^2-5x=0$$

or $$x(2x-5)=0$$

$\therefore$ $$x=0 \quad \text{or} \quad 2x-5=0$$

i.e., $$x=0 \quad \text{or} \quad 2x=5$$

Then $$x=0 \quad \text{or} \quad x=2{\cdot}5.$$

Hence, the two roots of the given quadratic equation are 0 and 2·5.

EXERCISE 2 (b)

(Section A)

Solve the following equations, the domain of the variable being the set of real numbers in each case :

1. $x^2-25=0.$ 2. $16x^2-9=0.$ 3. $5x^2=9+x^2.$
4. $3x^2-48=0.$ 5. $(x-2)^2-16=0,$ 6. $5(x-3)^2=180.$
7. $x^2-3x=0.$ 8. $5x^2-9x=0.$ 9. $16x^2-24x=0.$
10. $2x^2-3ax=0.$ 11. $3x^2=15x.$ 12. $4x^2+9x=0.$

(Section B)

Find the roots of the following quadratic equations using factorization :

13. $(x-3)^2=4(x-3).$ 14. $(x+5)(x-5)=39.$ 15. $(2x-3)(x-1)=12-5x.$

16. $\frac{1}{5}(x^2-1)+\frac{1}{3}(1-2x^2)=x^2.$

17. $(6+x)(5-x)+(5+x)(6-x)=10.$

Example 5. Solve $x^2-8x+15=0,$ when $x\in R.$

Solution.
$$x^2-8x+15=0$$
or $$x^2-5x-3x+15=0$$
or $$x(x-5)-3(x-5)=0$$
or $$(x-5)(x-3)=0$$

Split the middle term of the quadratic polynomial
$$15=(-5)(-3)$$
$$(-5)+(-3)=-8$$

$\therefore$ $x-5=0$ or $x-3=0$

Then $x=5$ or $x=3$

Hence the two roots of the given quadratic equation are 3 and 5.

Check :

When $x=5$
$$5^2-8\times5+15=0$$
$$25-40+15=0$$
$$40-40=0$$
$$0=0$$

When $x=3$
$$3^2-8\times3+15=0$$
$$9-24+15=0$$
$$24-24=0$$
$$0=0$$

Example 6. Solve $9x^2+15x-14=0,\ x\in R.$

Solution. $9x^2+15x-14=0$
or $9x^2+21x-6x-14=0$
or $3x(3x+7)-2(3x+7)=0$
or $(3x+7)(3x-2)=0$

$$9\times(-14)=-126$$
$$-126=21\times(-6)$$
$$21+(-6)=15$$

$\therefore$ $3x+7=0$ or $3x-2=0$

Then $x=-\frac{7}{3}$ or $x=\frac{2}{3}$

Thus, the two roots are $\frac{2}{3}$ and $-\frac{7}{3}.$

(Perform a check).

EXERCISE 2 (c)

Solve the following equations, when $x\in R$ in each case :

(Section A)

1. $x^2+6x-16=0$ 2. $x^2-7x+12=0$
3. $x^2+8x+15=0$ 4. $x^2-4x-5=0.$

(Section B)

5. $4x^2+4x-35=0$ 6. $3x^2+11x+10=0$ 7. $6x^2+x-15=0$
8. $12x^2-x-6=0$ 9. $3x^2+13x-10=0$ 10. $3x^2-10x=8$.

(Section C)

11. $(x+3)(5x+1)=3(x-1)$ 12. $2(2x+1)(x-1)=(2x-3)(4x-1)$.

2·4. SOLVING A QUADRATIC EQUATION BY COMPLETION OF SQUARES

Sometimes the polynomial of the given quadratic equation *cannot* be factorised by the method of splitting the middle term. Then we factorise it by the **method of completing the square.**

The method is illustrated by the following example :

Example 7. Solve $x^2+6x+2=0$, $x \in R$.

Solution. We cannot break up 2 into factors whose sum is 6. So the polynomial x^2+6x+2 cannot be factorised.

$$x^2+6x+2=0$$
or $$x^2+6x=-2$$
or $$x^2+2.3.x+3^2=-2+3^2$$
or $$(x+3)^2=7$$
$\therefore$ $$x+3=\pm\sqrt{7}$$
or $$x=-3\pm\sqrt{7}$$
Then $x=-3+\sqrt{7}$ or $x=-3-\sqrt{7}$

Hence the two roots are $-3+7$ and $-3-\sqrt{7}$.

EXERCISE 2 (d)

(Section A)

Solve the following quadratic equations by completing the squares, the domain of the variable being R:

1. $x^2-2x-1=0$. 2. $x^2+3x+1=0$. 3. $x^2+x=7$.
4. $x^2-6x-65=0$. 5. $x^2+3x+1=0$. 6. $3x^2-4x-60=0$.

(Section B)

Solve the following equations, giving your answer correct to two decimal places :

7. $2x^2-8x+5=0$. 8. $2x^2-3x-7=0$. 9. $3x^2-7x+1=0$.
10. $x^2-7x-5=0$. 11. $4x^2-11x=2$. 12. $3x^2-5x-4=0$.

2·5. ROOTS OF A QUADRATIC EQUATION AND THEIR NATURE

Let the quadratic equation be

$$ax^2+bx+c=0,\ a\neq 0 \qquad \ldots(1)$$

Multiplying both sides by $4a$, we get

$$4a^2x^2+4abx+4ac=0 \qquad \ldots(2)$$

α is a root of equation (1), if it is a root of equation (2) and *vice versa*.

So, it is enough to solve equation (2) which can be re-written as

$$(2ax)^2+2(2ax)(b)=-4ac$$
or $$(2ax)^2+2(2ax)(b)+(b)^2=b^2-4ac$$
$\therefore$ $$(2ax+b)^2=b^2-4ac \qquad \ldots(3)$$

Since α is a root of equation (2), it is also a root of equation (3).

When we substitute $x=\alpha$ on the L.H.S. of equation (3), then it becomes the square of a real number.

$\therefore$ L.H.S. $\geqslant 0$

Hence R.H.S. $\geqslant 0$ *i.e.*, $b^2-4ac \geqslant 0$.

What do you observe ?

If the quadratic equation : $ax^2+bx+c=0$ has a real root, then b^2-4ac must be $\geqslant 0$.

If $b^2-4ac \geqslant 0$, then $\sqrt{b^2-4ac}$ is a real number. Then equation (4) gives

$$2ax+b=\sqrt{b^2-4ac} \quad \text{or} \quad -\sqrt{b^2-4ac}$$

or

$$2ax=-b+\sqrt{b^2-4ac} \quad \text{or} \quad -b-\sqrt{b^2-4ac}$$

$\therefore$

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{or} \quad \frac{-b-\sqrt{b^2-4ac}}{2a}$$

What do you infer ?

If $b^2-4ac \geqslant 0$, then the quadratic equation $ax^2+bx+c=0$ has *two real roots* given by

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}, \qquad \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

The quantity b^2-4ac is called the **discriminant** of the quadratic equation $ax^2+bx+c=0$ and is denoted by D.

If $D=b^2-4ac$ is *zero*, then the roots

$\alpha=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$ are both equal to $-\dfrac{b}{2a}$. We, then say that this is a *repeated root*.

If $D=b^2-4ac$ is < 0, then the R.H.S. of equation (3) is negative. So, the L.H.S. should also be negative. But no real number α can be found which will make the L H.S. of equation (3) negative by substituting $x=\alpha$ on it.

What do you observe ?

If $D < 0$, then the quadratic equation $ax^2+bx+c=0$ has *no real roots.*

Let us summarize above discussions as under :

If $ax^2+bx+c=0$, $a\neq 0$, then $D=b^2-4ac$ is called the *discriminant*.

(1) If $D > 0$, then there are *two distinct real roots* given by

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}, \qquad \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

(2) If $D=0$, then there is a *repeated real root* given by

$$\alpha=-\frac{b}{2a}.$$

(3) If $D < 0$, then there are *no real roots.*

Example 8. Determine whether the quadratic equation $6x^2-7x+2=0$ has real roots. If it has find them.

Solution. The given quadratic equation is

$$6x^2-7x+2=0$$

Here we have $\quad a=6, \quad b=-7 \quad c=2$

$$D=b^2-4ac=(-7)^2-4\times 6\times 2$$
$$=49-48 \qquad =1.$$

$\therefore$ $D > 0$. Hence the equation has two real roots.

These roots are given by

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}, \qquad \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\alpha=\frac{-(-7)+\sqrt{1}}{2\times 6}, \qquad \beta=\frac{-(-7)-\sqrt{1}}{2\times 6}$$

Then $\alpha=\frac{7+1}{12}$ or $\frac{2}{3}$, $\qquad \beta=\frac{7-1}{12}$ or $\frac{1}{2}$

Therefore, the two roots are $\frac{2}{3}$ and $\frac{1}{2}$.

Example 9. Find the value of k so that the equation $9x^2+3kx+4=0$ has a repeated root.

Solution. The given quadratic equation is

$$9x^2+3kx+4=0$$

Here, we have $\quad a=9,\ b=3k,\ c=4$

$$D=b^2-4ac=(3k)^2-4\times 9\times 4$$
$$=9k^2-144$$

Since the quadratic equation has a repeated root, its discriminant is zero *i.e.*, $D=0$.

Then $\quad 9k^2-144=0 \quad$ or $\quad k^2-16=0$

or $\quad k^2=16 \quad \therefore \quad k=\pm 4.$

EXERCISE 2 (e)

(Section A)

1. Write the discriminant of the following quadratic equations :
   (*a*) $3x^2+2x-1=0$ (*b*) $2x^2-5x+3=0$
   (*c*) $x^2-\sqrt{3}x+2=0$ (*d*) $\sqrt{3}x^2-2\sqrt{2}x-2\sqrt{3}=0.$
2. Determine which of the following quadratic equations have real roots :
   (*a*) $x^2-4x-2=0$ (*b*) $2x^2-3x+1=0$
   (*c*) $3x^2-2\sqrt{2}x+1=0$ (*d*) $4x^2+12x+9=0.$
3. Determine which of the following quadratic equations have no real roots :
   (*a*) $x^2+4x+5=0$ (*b*) $2x^2+3=6x$
   (*c*) $2x^2-3x+6=0$ (*d*) $7x^2-8x=5.$

(Section B)

4. In the following determine the values of k for which the given quadratic equation has real roots :
   (*a*) $kx^2+4x+1=0$ (*b*) $x^2+4x+k=0$
   (*c*) $4x^2+2kx+9=0$ (*d*) $2x^2-kx+3=0.$
5. Show that each of the following quadratic equations has a repeated root and find that root :
   (*a*) $2x^2-2\sqrt{2}x+1=0$ (*b*) $3x^2+4\sqrt{3}x+4=0.$
6. In the following determine whether the given quadratic equations have real roots and if so find them :
   (*a*) $5x^2-4x+2=0$ (*b*) $2x^2+x-1=0$
   (*c*) $3x^2-5x+4=0$ (*d*) $6x^2-13x+6=0.$
7. Show that the following quadratic equations have two real roots and find these roots :
   (*a*) $2x^2-13x+20=0$ (*b*) $\sqrt{7}x^2-6x-13\sqrt{7}=0.$

(Section C)

8. In the following, find the value of k so that the given quadratic equation has equal roots :

(a) $2x^2+kx+8=0$ (b) $kx^2+8x=12$.

9. Show that the roots of $ax^2-2(a-b)\,x-4b=0$ are always real.

10. Determine the value of k such that the quadratic equation $x^2+7(3+2k)-2x(1+3k)=0$, has equal roots. [C.B.S.E., 1977 (Delhi)]

2·6. SUM AND PRODUCT OF ROOTS OF A QUADRATIC EQUATION

Let the quadratic equation be $ax^2+bx+c=0$, where $a\neq 0$.
Suppose its discriminant $D=b^2-4ac \geqslant 0$.
Then it has two real roots given by

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}, \qquad \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$\therefore$ The sum of the roots $=\alpha+\beta$

$$=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-2b}{2a} \quad =-\frac{b}{a}.$$

The product of the roots $=\alpha\beta$

$$=\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$=\frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2}$$

$$=\frac{b^2-(b^2-4ac)}{4a^2}=\frac{b^2-b^2+4ac}{4a^2}$$

$$=\frac{4ac}{4a^2}=\frac{c}{a}$$

Thus, $\quad \alpha+\beta=-\dfrac{\mathbf{b}}{\mathbf{a}}$ and $\alpha\beta=\dfrac{\mathbf{c}}{\mathbf{a}}$.

Example 10. Find the sum and product of the roots of the quadratic equation $4x^2-7x+5=0$.

Solution. The given quadratic equation is

$$4x^2-7x+5=0$$

Here $a=4$, $b=-7$, $c=5$

Sum of the roots $=-\dfrac{b}{a}=-\dfrac{-7}{4}$ *i.e.,* $\dfrac{7}{4}$

Product of the roots $=\dfrac{c}{a}=\dfrac{5}{4}$.

Example 11. Form a quadratic equation when sum of its roots is -3 and their product is 5.

Solution. Let the required quadratic equation be $ax^2+bx+c=0$.

Sum of the roots $=-\frac{b}{a}=-3$ (given)

$\therefore$ $b=3a$

Product of the roots $=\frac{c}{a}=5$ (given)

$\therefore$ $c=5a$

Then, the quadratic equation becomes

$$ax^2+3ax+5a=0$$

Dividing both sides by a, we get

$$x^2+3x+5=0$$

which is the required quadratic equation having sum of the root$=-3$ and product of the roots$=5$.

The above equation can be rewritten as

$$x^2-(-3)x+5=0$$

or $x^2-(\text{sum of roots})\,x+(\text{product of roots})=0.$

What do you observe ?

If sum of roots and product of roots are known, then a quadratic equation having these roots can be formed. Hence, if the roots are given separately, we can always find the quadratic equation corresponding to them.

Example 12. Find the quadratic equation whose roots are $1+\sqrt{5}$ and $1-\sqrt{5}$.
[*C.B.S.E., 1979 (A.I.)*]

Solution. Given roots are $1+\sqrt{5}$ and $1-\sqrt{5}$.

Sum of the roots $=1+\sqrt{5}+1-\sqrt{5}$
$=2$

Product of the roots $=(1+\sqrt{5})(1-\sqrt{5})$
$=1-5 \quad =-4$

$\therefore$ The required quadratic equation is

$$x^2-(\text{sum of the roots})\,x+(\text{product of the roots})=0$$

or $x^2-(2)x+(-4)=0$

i.e., $x^2-2x-4=0.$

EXERCISE 2 (f)

(Section A)

Find the sum and product of the roots of the following quadratic equations :

1. $x^2-2x+1=0.$
2. $x^2+3x-5=0.$
3. $4x^2+5x-2=0.$
4. $3x^2-4x+1=0.$
5. $x^2+px+q=0.$
6. $px-qx+r=0.$

(Section B)

Construct a quadratic equation whose roots have the sum and the product as under :

7. sum$=2$, product$=2$.
8. sum$=4$, product$=5$.
9. sum$=-3$, product$=4$.
10. sum$=-\frac{1}{6}$, product$=-\frac{1}{3}$.

Construct a quadratic equation whose roots are

11. -2 and 5 **12.** $-\frac{1}{2}$ and $-\frac{3}{2}$

13. $5+\sqrt{3}$ and $5-\sqrt{3}$ **14.** $\frac{2+\sqrt{5}}{3}$ and $\frac{2-\sqrt{5}}{3}$

15. Find the quadratic equation whose roots are $2\sqrt{3}, -2\sqrt{3}$.

[*C.B.S.E., 1980 (A.I.)*]

(Section C)

16. If one of the roots of the quadratic equation $2x^2+px+4=0$ is 2, find the other root. Also find the value of p.

17. One root of the quadratic equation $2x^2-5x+k=0$ is 3. Find the value of k and also the other root.

18. Find the value of k so that the sum of the roots of the equation $3x^2+(2k+1)\,x-k-5=0$ is equal to the product of the roots.

2·7. SYMMETRIC FUNCTIONS OF ROOTS

Consider the following expressions involving α and β, the roots of a quadratic equation $ax^2+bx+c=0$.

$$\alpha+\beta, \qquad \alpha^2+\beta^2, \qquad \alpha^3+\beta^3,$$

$$\alpha^2+\alpha\beta+\beta^2, \qquad \alpha^2-\alpha\beta+\beta^2, \qquad \frac{\alpha}{\beta}+\frac{\beta}{\alpha}$$

In each case interchange α and β and compare the new form with the original one. Are they different ?

All these expressions are symmetric in α and β.

An expression involving α and β, the roots of a quadratic equation is called a **symmetric function** of α and β, when it remains unchanged by interchanging α and β.

Now we shall find the values of these symmetric functions in terms of the co-efficients of the quadratic equation. Since we directly find the values of $\alpha+\beta$ and $\alpha\beta$, every symmetric function must be expressed in terms of $\alpha+\beta$ and $\alpha\beta$.

Example 13. If α, β are the roots of the equation $2x^2+3x-5=0$, find the value of $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$.

Solution. The given quadratic equation is $2x^2+3x-5=0$.

If α, β are the roots of the equation, then

$$\alpha+\beta=-\frac{b}{a}=-\frac{3}{2} \qquad [\text{Here } a=2, b=3, c=-5$$

$$\alpha\beta=\frac{c}{a}=-\frac{5}{2}$$

Now $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}$

$$=\frac{(\alpha+\beta)^2-2\,(\alpha\beta)}{\alpha\beta}$$

$$=\frac{\left(-\frac{3}{2}\right)^2-2\left(-\frac{5}{2}\right)}{-\frac{5}{2}}=\frac{\frac{9}{4}+5}{-\frac{5}{2}}$$

$$=\frac{\frac{29}{4}}{-\frac{5}{2}}=-\frac{29}{4}\times\frac{2}{5}$$

$$=-\frac{29}{10}\quad=-2{\cdot}9.$$

EXERCISE 2 (g)

(Section A)

1. If α, β are the roots of the equation $x^2-2x+3=0$, find the values of

(a) $\alpha^2\beta+\alpha\beta^2$ (b) $\frac{1}{\alpha}+\frac{1}{\beta}$ (c) $\alpha^2+\beta^2$

(Section B)

2. If α, β are the roots of the equation $x^2-px+q=0$, find the values of

(a) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ (b) $\alpha^3\beta+\alpha\beta^3$ (c) $\alpha^3+\beta^3$

(Section C)

3. Find the values of the following expressions, if α and β are the roots of the equation $ax^2+bx+c=0$.

(a) $(\alpha+1)(\beta+1)$ (b) $\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)^2$ (c) $\frac{1}{\alpha^3}+\frac{1}{\beta^3}$

2·8. FACTORIZATION OF QUADRATIC POLYNOMIALS

Let us consider a quadratic polynomial $p(x)=ax^2+bx+c$.

Let α be a root of the corresponding quadratic equation $p(x)=0$ *i.e.*, $ax^2+bx+c=0$.

Then $p(\alpha)=0$ and therefore by factor theorem $x-\alpha$ is a factor $p(x)$.

Thus, if α and β are roots of the quadratic equation $p(x)=0$, then $(x-\alpha)(x-\beta)$ is a factor of $p(x)$.

$\therefore$ $p(x)\equiv k(x-\alpha)(x-\beta)$, where k is a non-zero real number.

So, $ax^2+bx+c=k(x-\alpha)(x-\beta)$

Comparing the coefficients of x^2 on both sides, we get $k=a$

Then $p(x)\equiv a(x-\alpha)(x-\beta)$

Thus, if α and β are the roots of the quadratic equation $ax^2+bx+c=0$, the quadratic polynomial ax^2+bx+c can be factorised as $a(x-\alpha)(x-\beta)$.

If $b^2-4ac\geqslant 0$, the quadratic equation $ax^2+bx+c=0$ has two real roots α and β where

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}\text{ and }\beta=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Hence the quadratic polynomial ax^2+bx+c has the factorisation $a(x-\alpha)(x-\beta)$, where

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}\text{ and }\beta=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

If $b^2-4ac=0$, the quadratic equation $ax^2+bx+c=0$ has a real repeated root α, where $\alpha=-\frac{b}{2a}$. Hence the quadratic polynomial ax^2+bx+c has the factorisation $a(x-\alpha)^2$

i.e., $a\left(x-\frac{b}{2a}\right)^2$.

If $b^2-4ac < 0$, the quadratic equation $ax^2+bx+c=0$ has no real roots. Hence the quadratic polynomial ax^2+bx+c cannot be expressed as a product of two linear polynomials.

Example 14. Factorise the quadratic polynomial $\frac{1}{2}x^2-3x+4$ into linear factor over R, if possible.

Solution. The given polynomial is $\frac{1}{2}x^2-3x+4$.

Here $a=\frac{1}{2}$, $b=-3$, $c=4$.

Now $b^2-4ac=(-3)^2-4\left(\frac{1}{2}\right)(4)$

$=9-8 \qquad =1$

$\therefore \qquad b^2-4ac > 0.$

So, there exists a factorisation into real linear factors.

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}, \qquad \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{3+1}{2\times\frac{1}{2}} \qquad =\frac{3-1}{2\times\frac{1}{2}}$$

$\therefore \qquad \alpha=4, \qquad \beta=2.$

$\therefore$ The factorisation is $\frac{1}{2}(x-4)(x-2)$ *i.e.*, $(x-4)\left(\frac{1}{2}x-1\right)$.

Example 15. Factorise the quadratic polynomial $4\sqrt{3}x^2+5x-2\sqrt{3}$ into linear factors over R, if possible.

Solution. The given polynomial is $4\sqrt{3}x^2+5x-2\sqrt{3}$.

Here $a=4\sqrt{3}$, $b=5$, $c=-2\sqrt{3}$

Now $b^2-4ac=(5)^2-4(4\sqrt{3})(-2\sqrt{3})$

$=25+96 \qquad =121$

$\therefore \qquad b^2-4ac > 0.$

So, there exists a factorisation into real linear factors.

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}, \qquad \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-5+\sqrt{121}}{2\times4\sqrt{3}}, \qquad \beta=\frac{-5-\sqrt{121}}{2\times4\sqrt{3}},$$

$$=\frac{-5+11}{8\sqrt{3}}, \qquad =\frac{-5-11}{8\sqrt{3}}$$

$$\therefore \qquad \alpha=\frac{\sqrt{3}}{4},\ \beta=\frac{-2}{\sqrt{3}}$$

$\therefore$ The factorisation is $4\sqrt{3}\left(x-\frac{\sqrt{3}}{4}\right)\left(x+\frac{2}{\sqrt{3}}\right)$

i.e., $(4x-\sqrt{3})(\sqrt{3}x+2)$

EXERCISE 2 (h)

(Section A)

1. Determine which of the following quadratic polynomials can be factorised into a product of real linear factors :

(a) $3x^2+5x+2$ (b) $3x^2+2x+1$

(c) $2x^2-5x+7$ (d) $\sqrt{3}x^2+10x+8\sqrt{3}$

(Section B)

2. In the following, find the value(s) of k for which the quadratic has real linear factors :

(a) $2x^2+6x+k$ (b) kx^2-5x+2

(c) $3x^2-4x-2k$ (d) kx^2-4+3x

3. Factorise the following quadratic polynomials :

(a) $2x^2-3x+1$ (b) $2x^2+11x+5$

(c) $6x^2-5x-21$ (d) $\frac{1}{2}x^2-3x+4$

(Section C)

4. Factorise the following quadratic polynomials into linear factors over R, if possible :

(a) x^2+4x+2 (b) $2x^2+3x-7$

(c) $x^2+10x-2$ (d) $3x^2+6x-2$

2·9. EQUATIONS REDUCIBLE TO QUADRATIC EQUATIONS

Sometimes we have to solve equations which, though not quadratic, can be reduced to quadratic equations by making suitable substitutions. We shall call such equations as equations reducible to quadratic equations. We shall now deal with several types of such equations.

Type 1. $\mathbf{ax^4+bx^2+c=0}$.

The equation involves fourth degree polynomial having only even powers of x. Putting $x^2=z$, the equation reduces to $az^2+bz+c=0$ which being a quadratic in z can be solved.

Example 16. Solve $4x^4-25x^2+36=0$.

Solution. The given equation is $4x^4-25x^2+36=0$

Substituting $z=x^2$, we get

$$4z^2-25z+36=0$$

or $4z^2-16z-9z+36=0$

or $4z(z-4)-9(z-4)=0$

or $(z-4)(4z-9)=0$

If $z-4=0$, then $z=4$

If $4z-9=0$, then $z=\frac{9}{4}$.

Now substituting $z=x^2$, we get

$x^2=4$, $x^2=\frac{9}{4}$.

$\therefore\ x=\pm 2$ $\therefore\ x=\pm\frac{3}{2}$.

Let us verify these solutions of the given equation in x.

When $x=2$, L.H.S. $=4\times 2^4-25\times 2^2+36$
$=64-100+36$ $=0$
$=$R.H.S.

When $x=-2$, L.H.S. $=4(-2)^4-25(-2)^2+36$
$=64-100+36$ $=0$
$=$R.H.S.

When $x=\frac{3}{2}$, L.H.S. $=4\left(\frac{3}{2}\right)^4-25\left(\frac{3}{2}\right)^2+36$

$=\frac{81}{4}-\frac{225}{4}+36$ $=0$

When $x = -\frac{3}{2}$, L.H.S.$=4\left(-\frac{3}{2}\right)^4-25\left(-\frac{3}{2}\right)^2+36$

$$=\frac{81}{4}-\frac{225}{4}+36 \qquad =0$$

$$=\text{R.H S.}$$

Thus, the solutions of the given equation are

$$x=\pm 2 \text{ and } x=\pm\frac{3}{2}.$$

Type 2. $\mathbf{py+\frac{q}{y}=r}$.

Here y occurs in the denominator of one of the terms in the equation. So, we must seek those real values of y which are not zero and satisfy the given equation. By multiplying both sides by y, the equation reduces to $py^2+q=ry$ or $py^2-ry+q=0$. This is a quadratic equation in y and can be solved for y.

Example 17. Solve $2x+\frac{4}{x}=9$.

Solution. The given equation is $2x+\frac{4}{x}=9$

Multiplying both sides by x, we get

$$2x^2+4=9x, \qquad \text{when } x\neq 0$$

or $$2x^2-9x+4=0$$

The solutions are

$$x=\frac{-(-9)\pm\sqrt{(-9)^2-4\times 2\times 4}}{2\times 2}=\frac{9\pm\sqrt{81-32}}{4}$$

Then $$x=\frac{9\pm 7}{4} \qquad \text{i.e., } 4,\ \frac{1}{2}$$

Thus, the solutions of the equation are $x=4$ and $x=\frac{1}{2}$.

Type 3. $\mathbf{\sqrt{a-x^2}=bx+c}$.

This equation involves only one radical. Here we have to seek those solutions for which $a-x^2\geqslant 0$ *i.e.*, $x^2\leqslant a$ and R.H.S. $\geqslant 0$ *i.e.*, $bx+c\geqslant 0$. Thus, we need solutions for which $x^2\leqslant a$ and $bx+c\geqslant 0$.

Squaring both sides of the equation, we get

$$a-x^2=(bx+c)^2$$

or $$a-x^2=b^2x^2+2(bx)c+c^2$$

or $$(b^2+1)x^2+2bcx+(c^2-a)=0.$$

It is a quadratic equation which can be solved for x, where $x^2\leqslant a$ and $bx+c\geqslant 0$.

Example 18. Solve $x-\sqrt{25-x^2}=1$.

Solution. The given equation is $x-\sqrt{25-x^2}=1$.

It can be written as

$$x-1=\sqrt{25-x^2}$$

Here we have to seek solutions for which $25-x^2\geqslant 0$ *i.e.*, $x^2\leqslant 25$ and $x-1\geqslant 0$ *i.e.*, $x\geqslant 1$.

Now squaring both sides of the equation, we get

$$x^2-2x+1=25-x^2$$

or $$2x^2-2x-24=0$$

or $$x^2-x-12=0$$

The roots of the equation are

$$x=\frac{-(-1)\pm\sqrt{(-1)^2-4\times1\times(-12)}}{2\times1}$$

$$=\frac{1\pm\sqrt{1+48}}{2}=\frac{1\pm7}{2}$$

$\therefore$ $$x=4,\ -3.$$

These roots must satisfy both the conditions, $x^2 \leqslant 25$ and $x \geqslant 1$.

We find that $x=4$ is the only solution of the given equation, satisfying both the conditions.

EXERCISE 2 (i)

(Section A)

Solve the following equations by reducing them to quadratic equations :

1. $x^4-13x^2+36=0$.
2. $25x^4-20x^2+4=0$.
3. $2x^4-5x^2+3=0$.
4. $9x^4-148x^2+64=0$.
5. $2x-\dfrac{3}{x}=1$.
6. $3x+\dfrac{5}{16x}=2$.
7. $\sqrt{3x^2-2}=2x-1$.
8. $\sqrt{13-x^2}=x+5$.

(Section B)

Determine the real solutions of the following equations :

9. $3x-5\sqrt{x}+2=0$.
10. $\sqrt{x}+2x=1$.
11. $8x^6-91x^3+216=0$.
12. $x^{2/3}+12=7x^{1/3}$.
13. $x+\sqrt{x-2}=8$.
14. $x-\sqrt{3x-6}=2$.

[*C.B.S.E., 1979 (A.I.)*]

(Section C)

15. $3^{2x}-10.3^x+9=0$.
16. $4^x-36\cdot2^x+128=0$.

Type 4. $\sqrt{\mathbf{ax+b}}\pm\sqrt{\mathbf{cx+d}}=\mathbf{e}$

This equation involves two radicals. Here we have to seek those solutions for which $ax+b \geqslant 0$ and $cx+d \geqslant 0$. The following solved example will help you to understand the method of solving such equations.

Example 19. Solve $\sqrt{2x+9}-\sqrt{x-4}=3$.

Solution. The given equation is

$$\sqrt{2x+9}-\sqrt{x-4}=3$$

Here we must look for solutions which satisfy

$$2x+9 \geqslant 0 \quad i.e., \quad x \geqslant -\frac{9}{2} \qquad ...(1)$$

$$x-4 \geqslant 0 \quad i.e., \quad x \geqslant 4 \qquad ...(2)$$

In order to satisfy both conditions (1) and (2), we must have $x \geqslant 4$.

The equation can be written as :

$$\sqrt{2x+9}=3+\sqrt{x-4}.$$

Squaring both the sides, we get

$$2x+9=9+6\sqrt{x-4}+x-4$$

Regrouping the terms so that the radical is one side and all other terms are on the other side, we get

$$2x-x+9-9+4=6\sqrt{x-4}$$

or $$x+4=6\sqrt{x-4}$$

Squaring both sides again, we get

$$x^2+8x+16=36\ (x-4)$$

or $$x^2+8x+16=36x-144$$

or $$x^2+8x-36x+16+144=0$$

or $$x^2-28x+160=0$$

or $$x^2-20x-8x+160=0$$

or $$x(x-20)-8(x-20)=0$$

or $$(x-20)(x-8)=0$$

If $x-20=0$, then $x=20$

If $x-8=0$, then $x=8$.

Thus, the roots of the equation are $x=8$ and $x=20$.

Since both roots satisfy the above condition $x \geqslant 4$, both are the solutions of the given equation. Hence, the required solutions are $x=8$, $x=20$.

Type 5. $\mathbf{a}\left(\mathbf{x}^2+\frac{\mathbf{1}}{\mathbf{x}^2}\right)+\mathbf{b}\left(\mathbf{x}+\frac{\mathbf{1}}{\mathbf{x}}\right)+\mathbf{c}=\mathbf{0}.$

This equation can be rewritten as a quadratic equation in terms of $x+\frac{1}{x}$. On solving it, we get Type 2 form discussed before.

Example 20. Solve $2\left(x^2+\frac{1}{x^2}\right)-3\left(x+\frac{1}{x}\right)-1=0,\ x\neq 0.$

Solution. The given equation is

$$2\left(x^2+\frac{1}{x^2}\right)-3\left(x+\frac{1}{x}\right)-1=0.$$

Let $x+\frac{1}{x}=y$ Then $x^2+\frac{1}{x^2}=y^2-2.$

Substituting these in the given equation, we have

$$2(y^2-2)-3y-1=0$$

or $$2y^2-3y-5=0.$$

The roots of the equation are

$$y=\frac{-(-3)\pm\sqrt{9-4\times 2\times(-5)}}{2\times 2} \quad i.e., \quad y=\frac{3\pm\sqrt{9+40}}{4}$$

$$\therefore \quad y=\frac{3+7}{4}, \quad y=\frac{3-7}{4} \quad i.e., \quad y=\frac{5}{2},\ y=-1$$

Now $$y=x+\frac{1}{x} \qquad \therefore \quad x+\frac{1}{x}=\frac{5}{2}$$

or $$2x^2+2=5x \quad \text{or} \quad 2x^2-5x+2=0$$

$$\therefore \quad x=\frac{-(-5)\pm\sqrt{25-4\times 2\times 2}}{2\times 2}=\frac{5\pm\ 9}{4}$$

Then $x=\frac{5+3}{4}$, $x=\frac{5-3}{4}$ *i.e.*, $x=2, x=\frac{1}{2}$

Again $y=x+\frac{1}{x}$ $\therefore$ $x+\frac{1}{x}=-1$

or $x^2+1=-x$ or $x^2+x+1=0$

Here $b^2-4ac=1-4\times1\times1=-3$.

Since $b^2-4ac<0$, this equation has no real roots.

Thus, the solutions of the given equation are

$$x=2,\ x=\tfrac{1}{2}$$

We give below more examples of equations reducible to quadratic equations.

Example 21. Solve $\left(\frac{2x-3}{x-1}\right)-4\left(\frac{x-1}{2x-3}\right)=3,\ x\neq1,\ x\neq\frac{3}{2}$.

[*C.B.S.E., 1978,* (*A.I.*)]

Solution. The given equation is

$$\left(\frac{2x-3}{x-1}\right)-4\left(\frac{x-1}{2x-3}\right)=3.$$

Substituting $\frac{2x-3}{x-1}=y$ in the equation, we have

$$y-4\times\frac{1}{y}=3$$

or $y^2-4=3y$ or $y^2-3y-4=0$

The roots of the equation are

$$y=\frac{-(-3)\pm\sqrt{9-4\times1\times(-4)}}{2\times1}=\frac{3\pm\sqrt{9+16}}{2}$$

$\therefore$ $y=\frac{3+5}{2}$, $y=\frac{3-5}{2}$ *i.e.*, $y=4,\ y=-1$

Now $y=\frac{2x-3}{x-1}$ $\therefore$ $\frac{2x-3}{x-1}=4$

or $2x-3=4x-4$ or $2x-4x=-4+3$

or $-2x=-1$ or $2x=1$ $\therefore$ $x=\frac{1}{2}$

Also $\frac{2x-3}{x-1}=-1$ or $2x-3=-x+1$

or $2x+x=1+3$ or $3x=4$ $\therefore$ $x=\frac{4}{3}$

Hence the solutions of the given equation are

$$x=\frac{1}{2},\ x=\frac{4}{3}.$$

Example 22. Solve $7^{1+x}+7^{1-x}=50$

Solution. The given equation is

$$7^{1+x}+7^{1-x}=50$$

or $\quad 7 \cdot 7^x + 7 \cdot 7^{-x} = 50 \quad$ or $\quad 7 \cdot 7^x + 7 \cdot \frac{1}{7^x} = 50$

Substituting $7^x = y$, we get

$$7y + \frac{7}{y} = 50 \quad \text{or} \quad 7y^2 + 7 = 50y$$

or $\quad 7y^2 - 50y + 7 = 0$

or $\quad 7y^2 - 49y - y + 7 = 0$

or $\quad 7y(y-7) - 1(y-7) = 0$

or $\quad (y-7)(7y-1) = 0$

If $\quad y-7=0, \quad$ then $y=7$,

If $\quad 7y-1=0, \quad$ then $y=\frac{1}{7}$

Now $\quad y=7, \quad y=\frac{1}{7}$

$\therefore \quad 7^x=7^1, \quad 7^x=7^{-1}$

Since the bases are same on both sides of the equation, so equating their exponents, we get $\quad x=1, \quad x=-1$

Hence the solutions of the given equation are

$$x=1, \qquad x=-1$$

EXERCISE 2 (j)

(Section A)

Solve the following by reducing them to quadratic equations :

1. $\sqrt{x+4} + \sqrt{x-5} = 3$
2. $\sqrt{1-5x} + \sqrt{1-3x} = 2$
3. $\sqrt{2x+1} + \sqrt{x+4} = 3$
4. $\sqrt{2x+3} - \sqrt{x+1} = 1$
5. $\left(x^2+\frac{1}{x^2}\right)+4\left(x+\frac{1}{x}\right)+6=0, \quad x\neq 0$
6. $9\left(x^2+\frac{1}{x^2}\right)-27\left(x+\frac{1}{x}\right)+8=0, \quad x\neq 0$
7. $3^{x-1}+3^{1-x}=2$
8. $5^{1+x}+5^{1-x}=26$

(Section B)

Solve the following equations :

9. $\left(\frac{2x+1}{x-1}\right)+\left(\frac{x-1}{2x+1}\right)=2\frac{1}{2}, \quad x\neq 1, \quad x\neq -\frac{1}{2}.$
10. $\left(\frac{2x-1}{x+1}\right)-15\left(\frac{x+1}{2x-1}\right)=-2, \quad x\neq -1, \quad x\neq \frac{1}{2}.$ [*C.B.S.E., 1986 (A.I.)*]
11. $2\left(x^2+\frac{1}{x^2}\right)+\left(x-\frac{1}{x}\right)-10=0, \quad x\neq 0.$
12. $8\sqrt{\frac{x}{x+3}}-\sqrt{\frac{x+3}{x}}=2, \quad x\neq 0, \quad x\neq -3.$ [*C.B.S.E., 1986 (Delhi)*]

13. $\left(x^2+\frac{1}{x^2}\right)-3\left(x-\frac{1}{x}\right)-2=0, \quad x\neq 0.$

14. $\sqrt{\frac{x}{1-x}}+\sqrt{\frac{1-x}{x}}=2\frac{1}{6}, \ x\neq 0, \ x\neq 1.$

(Section C)

Solve for x :

15. $\sqrt{(x-1)(x-2)}+\sqrt{(x-3)(x-4)}=2$

16. $\sqrt{2x^2+x+3}+\sqrt{2x^2+x-6}=3.$

2·10. PROBLEMS INVOLVING QUADRATIC EQUATIONS

There are many word problems which can be solved by setting up a quadratic equation whose solution will be a solution to the problem. Sometimes only one root out of two roots of the quadratic equation has a meaning for the problem. Any root of the quadratic equation *not* satisfying the conditions of the given problem must be *rejected.*

The method of solving verbal problems leading to quadratic equations consists of three steps :

(1) *translating the word problem into the symbolic language,*

(2) *solving the equation, and*

(3) *interpreting the solution of the equation.*

The following example will illustrate the method :

Example 23. The difference of two positive integers is 8 and the sum of their squares is 274. Find the numbers.

Solution. Let one number be x.

Then the other number$=x+8$

Sum of their squares$=274$

Then $x^2+(x+8)^2=274$

or $x^2+x^2+16x+64=274$

or $2x^2+16x-210=0$

or $x^2+8x-105=0$

or $x^2+15x-7x-105=0$

or $x(x+15)-7(x+15)=0$

or $(x+15)(x-7)=0$

If $x+15=0,$ then $x=-15.$

If $x-7=0,$ then $x=7.$

Rejecting the negative value as given numbers are positive integers, we get

$x=7$

One number is 7. Then the other number is $7+8$, *i.e.*, 15.

So required numbers are 7 and 15.

Example 24. A man covers a distance of 200 *km* travelling with a uniform speed of 'x' *km* per hour. The distance could have been covered in 2 hours less had the speed been $(x+5)$ *km* per hour. Calculate the value of x.

Solution. In first case, the time taken by the man to cover 200 *km* at x *km* per

hour$=\frac{200}{x}$ hours.

In second case, the time taken by the man to cover 200 km at $(x+5)$ km per hour $= \dfrac{200}{x+5}$ hours

According to the given condition,

$$\frac{200}{x}-\frac{200}{x+5}=2$$

$$\text{or} \quad \frac{100}{x}-\frac{100}{x+5}=1$$

$$\text{or} \quad 100(x+5)-100x=x(x+5)$$

$$\text{or} \quad 100x+500-100x=x^2+5x$$

$$\text{or} \quad x^2+5x-500=0$$

$$\text{or} \quad x^2+25x-20x-500=0$$

$$\text{or} \quad x(x+25)-20(x+25)=0$$

$$\text{or} \quad (x+25)(x-20)=0$$

$$x=-25,\ 20.$$

Rejecting the negative value, we get the value of x as 20 km/hour.

EXERCISE 2 (k)

Numbers :

1. Thrice the square of a number is 243. Find the number.
2. The product of two consecutive odd numbers is 399. Find the numbers.
3. Find two consecutive even numbers such that the sum of their squares is 100.
4. The sum of two numbers is 25 and the sum of their squares is 313. Find the numbers.
5. The sum of the squares of two consecutive positive integers is 545. Find the numbers.
6. The sum of the squares of three consecutive natural numbers is 194. Find them.
7. The product of two successive multiples of 3 is 180. Determine the multiples.
8. Find a possible number which when decreased by 20 is equal to 69 times the reciprocal of the number.
9. Find two natural numbers which differ by 3 and the sum of whose squares is 117.
10. Five times a certain whole number is equal to three less twice the square of the number. Find the number.
11. A body travels at x metres per minute for 8 minutes and then for another $4x$ minutes at the same speed. If the total distance covered is 117 metres, find the speed of the body.
12. Divide 41 into two parts whose product is 288.
13. Divide 15 into two parts such that the sum of their squares is 113.
14. The sum of a number and its reciprocal is $2\frac{4}{15}$. What is the number ?

Area :

15. The sides of a right-angled triangle containing the right angle are $5x$ cm and $(3x-1)$ cm. If the area of the triangle be 60 cm^2, calculate the lengths of the sides of the triangle.
16. Let x denote the breadth of a rectangle whose length exceeds the breadth by 3 units. If the numerical values of the area and the perimeter of the rectangle are equal, find x. [*C.B.S.E., 1978 (A.I.)*]

17. A rectangle has an area of 24 cm^2. If its length is x *cm*, write down its breadth in terms of x. Given that its perimeter is 20 *cm*, form an equation in x and solve it. Also find the dimensions of the rectangle.

18. The perimeter of a rectangle is 82 *m* and its area is 400 m^2. Find the breadth of the rectangle. [*C.B.S.E., 1981 (Delhi)*]

19. The area of a right triangle is 66 *sq. cm*. If the base of the triangle exceeds that of its altitude by 5 cm, find the altitude of the triangle. [*C.B.S.E., 1984 (A.I.)*]

20. The hypotenuse of a right-triangle is 13 *cm* and the difference between the other two sides is 7 *cm*.

(*i*) Taking 'x' as the length of the shorter of the two sides, write an equation in 'x' that represents the above statement.

(*ii*) Solve the equation obtained in (*i*) above, and hence find the two unknown sides of the triangle.

21. The length of a verandah is 3 *m* more than its breadth. The numerical value of its area is equal to the numerical value of its perimeter.

(*i*) Taking 'x' as the breadth of the verandah, write an equation in 'x' that represents the above statement.

(*ii*) Solve the equation obtained in (*i*) above and hence find the dimensions of the verandh.

Mixed:

22. If a cyclist had gone 3 *km* per hour faster, he would have taken 1 hour 20 minutes less to ride 80 *km*. What time did he take ?

23. AB is a segment line whose length is 6 *cm*. Find a point P in it such that

$$AB.PB = AP^2.$$

24. Three consecutive numbers are such that the square of the middle number exceeds the difference of the squares of the other two by 60. Assume the middle number to be x and form a quadratic equation satisfying the above statement. Hence find the three numbers.

25. The sum of the ages of Puneet and his father is 45 years and the product of their ages is 126. Find the ages of the father and the son.

26. The product of Ravi's age (in years) five years ago with his age (in years) 9 years later is 15. Find Ravi's present age.

27. A train covers a distance of 200 *km* between two stations at a speed of 'x' *km/hour*. Another train covers the same distance at a speed of $(x+5)$ *km/hour*.

(*i*) Find the time which each train takes to cover the distance between the stations.

(*ii*) If the second train takes two hours less than the first, find the value of 'x'.

28. A trader bought a number of articles for Rs. 1200. Ten were damaged and he sold each of the rest at Rs. 2 more than what he paid for it, thus clearing a profit of Rs. 60 on the whole transaction. Taking the number of articles he bought as x, form an equation in x and solve it.

29. Some students planned a picnic. The budget for food was Rs. 24. Because four of them failed to go, the cost of food to each member got increased by Re. 1. How many students attended the picnic ? [*C.B.S.E., 1983 (A.I.)*]

30. The length of the hypotenuse of a right triangle exceeds the length of the base by 2 *cm* and exceeds twice the length of the altitude by 1 *cm*. Find the length of each side of the triangle.

REVIEW EXERCISE II

(Section A)

1. Fill in the blanks to make the following statements true :

(*a*) D (or discriminant) of $-3x^2+2x-8=0$ is......... [*C.B.S.E.*, *1978* (*Delhi*)]

(*b*) The equation $x^2-px+q=0$; $p, q \in R$ has no real roots, if......... [*C.B.S.E.*, *1986* (*A.I.*)]

(*c*) A quadratic equation $px^2+qx+r=0, p\neq 0$ has equal roots, if......... [*C.B.S.E.*, *1985* (*Delhi*)]

(*d*) The quadratic equation with roots 5, -3 is......... [*C.B.S.E.*, *1987* (*Delhi*)]

(*e*) The product of the roots of the equation $x^2=5$ is......... [*C.B.S.E.*, *1986* (*Delhi*)]

(*f*) The equation whose roots are $1+\sqrt{3}$ and $1-\sqrt{3}$ is......... [*C.B.S.E.*, *1982* (*A.I.*)]

(*g*) Every quadratic polynomial can have at most.........zeros.

(*h*) The sum of the roots of the equation $2x^2-3x+5=0$ is.........

2. If α, β are the roots of the equation $3x^2+7x+3=0$, write down the value of :

(*a*) $\alpha+\beta$ (*b*) $\alpha\beta$. [*C.B.S.E.*, *1981* (*Delhi*)]

3. Find the equation whose roots are $3+\sqrt{2}$ and $3-\sqrt{2}$. [*C.B.S.E.*, *1983* (*Delhi*)]

4. Comment upon the nature of roots of the following :

(*a*) $5y^2+12y-9=0$

(*b*) $9a^2b^2x^2-48\,abcdx+64c^2d^2=0, a\neq 0, b\neq 0$ [*C.B.S.E.*, *1977* (*Delhi*)]

(Section B)

5. Use the quadratic formula to solve

$$15x^2-7x-36=0$$ [*C.B.S.E.*, *1979* (*Delhi*)]

6. Find the roots of the quadratic equation

$$48y^2-13y-1=0$$ [*C.B.S.E.*, *1980* (*A.I.*)]

7. Solve for y : $$\left(\frac{7y-1}{y}\right)^2-3\left(\frac{7y-1}{y}\right)-18=0,\ y\neq 0$$ [*C.B.S.E.*, *1985* (*Delhi*)]

8. Solve for y : $$6\left(\frac{y-3}{2y+1}\right)+1=5\sqrt{\frac{y-3}{2y+1}},\ y\neq -\frac{1}{2}.$$ [*C.B.S E.*, *1977* (*Delhi*)]

9. Solve the following equation :

$$\sqrt{\frac{x}{x-3}}+\sqrt{\frac{x-3}{x}}=2\frac{1}{2},\ x\neq 0,\ x\neq 3.$$ [*C.B.S.E.*, *1986* (*A.I.*)]

(Section C)

10. The sum of the squares of three consecutive natural numbers is 110. Determine the numbers. [*C.B.S.E.*, *1984* (*A.I.*)]

11. The length of a room is 3 *m.* more than its breadth. If the area of the room is 70 *sq. m*, determine the dimensions of the room.

12. A segment AB of 2 *m* length is divided at C into two parts such that $AC^2=AB\,.\,CB$. Find the length of part CB. [*C.B.S.E.*, *1980* (*Delhi*)]

13. If I had walked 1 *km* per hour faster, I would have taken 10 minutes less to walk 2 *km* Find the rate of my walking. [*C.B.S.E.*, *1982* (*A.I.*)]

14. A farmer wishes to start a 100 *sq. m.* rectangular vegetable garden. Since he has 30 *m.* barbed wire, he fences three sides of the rectangular garden letting his house compound wall act as the fourth side fence. Find the dimensions of his garden.

15. A cyclist cycles non-stop from A to B a distance of 14 *km* at a certain average speed. If his average speed reduces by 1 *km* per hour, he takes $\frac{1}{3}$ hour more to cover the same distance. Find his original average speed. [*C.B.S.E.*, *1986* (*Delhi*)]

□ □

3

RATIONAL EXPRESSIONS

3·1. In your previous class you have learnt about H.C.F. and L.C.M. of two polynomials. Remember that the H.C.F. is also called the **greatest common divisor**, written in short as **G.C.D.**

EXERCISE 3 (a)

1. Find the g.c.d. of the polynomials $(x-3)^2(x-2)(x+1)^2$ and $(x-3)(x+1)^2(x-4)$.
2. Find the g.c.d of the polynomials $4(x+3)^2(x-1)(x+1)^3$ and $6(x-1)^2(x+1)^2(x+7)$.
3. Find the g.c.d. of the two polynomials :
   $p(x)=x^2-5x+6$ and $q(x)=x^2+5x-14$.
4. Find the g.c.d. of the two polynomials :
   $p(x)=6x^2+11x+3$ and $q(x)=2x^2+x-3$
5. Find the g.c.d. of the following pair of polynomials :
   $24(6x^4-x^3-2x^2)$; $20(2x^6+3x^5+x^4)$.
6. Find the l.c.m. of the polynomials $(x+3)(x-2)^2$ and $(x-2)(x-6)$.
7. Find the l.c.m. of the two polynomials,
   $p(x)=x^2+7x+12$ and $q(x)=x^2+8x+16$.
8. Find the l.c.m. of the two polynomials,
   $p(x)=-x^2-x+6$ and $q(x)=-x^2+x+2$.
9. Find the l.c.m. of the following pair of polynomials :
   $(x+3)(-6x^2+5x+4)$; $(2x^2+7x-3)(x+3)$.
10. Find the l.c.m. of the polynomials :
    x^3-x^2-5x+2 and x^3+4x^2+x-6.

3·2. RATIONAL EXPRESSION

Let us recall the properties of integers and polynomials. Now compare the properties of integers with those of polynomials.

What do you observe ?

We observe that polynomials possess properties *exactly similar* to those of integers. So, we may say that polynomials behave like integers. We form rational expressions from polynomials just as we formed rational numbers from integers.

A rational number is the quotient $\frac{m}{n}$ of two integers m and n where $n \neq 0$. Similarly, if $p(x)$ and $q(x)$ are two polynomials, then $\frac{p(x)}{q(x)}$ need not be a polynomial, where $q(x)$ is a

non-zero polynomial. Then we say that $\frac{p(x)}{q(x)}$ is a **rational expression.**

We define a rational expression as the quotient $\frac{p(x)}{q(x)}$ of two polynomials $p(x)$ and $q(x)$, where $q(x)$ is not the zero polynomial.

In the rational expression $\frac{p(x)}{q(x)}$, $p(x)$ is called the **numerator**

, $q(x)$ is called the **denominator** ;

$p(x)$ and $q(x)$ are called the **terms** of the expression.

By this definition every integer, every rational number and, in general, every polynomial is also a rational expression :

$$3,\ \frac{1}{5},\ x^2,\ 5x^2+3,\ \frac{1}{x},\ \frac{x^2-2x+1}{x^3+1},\ \frac{x-\frac{1}{2}}{\frac{1}{2}x+\frac{3}{5}}$$

These are all examples of rational expressions.

EXERCISE 3 (b)

(Section A)

1. Which of the following are rational expressions ?

(*a*) $\frac{x^2-x+3}{x+5}$ (*b*) $\frac{x^2-2}{3\sqrt{x}+5}$

(*c*) $\frac{\sqrt{3x^2-4x+7}}{2x-\sqrt{6}}$ (*d*) $\frac{x^3+3x^2-1}{x^2+\sqrt{x}-2}$.

(Section B)

2. Write a rational expression whose numerator is a linear polynomial and denominator is a quadratic polynomial.
3. Write a rational expression whose numerator is a binomial and whose denominator is a trinomial.

(Section C)

4. Write a rational expression whose numerator is a quadratic polynomial whose zeros are 1 and -2 and whose denominator is a quadratic polynomial whose zeros are 2 and $\frac{1}{3}$.

3·3. RATIONAL EXPRESSIONS IN LOWEST TERMS

You know that a rational number $\frac{m}{n}$ is said to be in its lowest terms if g.c.d. of its numerator and denominator *i.e.*, m and n is 1. If $\frac{m}{n}$ is not in its lowest terms, we get it in the lowest terms by cancelling the g.c.d. from both the numerator and denominator. In the same way, *a rational expression is expressed in its* **lowest terms** *by cancelling out non-zero common factors (g.c.d.) of both the numerator and denominator.*

When a rational expression is written in its lowest terms, it is in its **simplest form.**

Example 1. Reduce to lowest terms : $\frac{x^2+x-2}{x^2-9x+18}$

Solution. Numerator$=x^2+x-2$

$=x^2+2x-x-2$

$=x(x+2)-1\,(x+2)=(x+2)(x-1)$

$$\text{Denominator} = x^2 - 9x + 18$$
$$= x^2 - 6x - 3x + 18$$
$$= x(x-6) - 3(x-6) \quad = (x-6)(x-3)$$

$\therefore$ g.c.d. of the numerator and denominator $= 1$.

Thus, the given rational expression is in lowest terms.

Example 2. Reduce the rational expression $\dfrac{4x^2-4}{2x^2+6x-8}$ to its lowest terms.

Solution. $p(x) = 4x^2 - 4 \quad = 4(x^2-1)$

$$= 4(x+1)(x-1)$$

$$q(x) = 2x^2 + 6x - 8 = 2(x^2+3x-4)$$
$$= 2[x^2+4x-x-4]$$
$$= 2[x(x+4) - 1(x+4)]$$
$$= 2(x+4)(x-1)$$

$\therefore$ g.c.d. of $p(x)$ and $q(x)$ is $2(x-1)$.

Now $\qquad \dfrac{4x^2-4}{2x^2+6x-8} = \dfrac{4(x+1)(x-1)}{2(x+4)(x-1)} \qquad = \dfrac{2(x+1)}{x+4},\ x \neq 1$

[Cancelling the g.c.d. $2(x-1)$, we get the simplest form.]

EXERCISE 3 (c)

(Section A)

Reduce the following rational expressions to their simplest form :

1. $\dfrac{(x+3)(x-2)}{(x-1)}$ **2.** $\dfrac{(x+2)(x-3)}{(x+1)(x-2)}$

3. $\dfrac{3x-3}{5x-5}$ **4.** $\dfrac{x^2-3x}{9x-x^3}$

5. $\dfrac{(x-3)(x+2)}{(x+2)(x+1)}$ **6.** $\dfrac{x^2-9}{2x^3-6x^2}$

(Section B)

Reduce the following rational expressions to their lowest terms :

7. $\dfrac{x^2-6x+8}{x^2-5x+6}$ **8.** $\dfrac{x^2-2x-3}{2x^2-5x-3}$

9. $\dfrac{6x^2-7x-5}{6x^2-19x+15}$ **10.** $\dfrac{15x^2-35x+10}{x^2-7x+10}$

11. $\dfrac{6x^2-7x-20}{9x^2+6x-8}$ **12.** $\dfrac{6x^2-7xy-5y^2}{6x^2-xy-15y^2}$

3·4. ADDITION OF RATIONAL EXPRESSIONS

If $\dfrac{p(x)}{q(x)}$ and $\dfrac{r(x)}{s(x)}$ are two rational expressions, we define their sum as,

$$\frac{p(x)}{q(x)} + \frac{r(x)}{s(x)} = \frac{p(x)s(x) + q(x)r(x)}{q(x)s(x)}.$$

The definition of addition implies that

$$\frac{p(x)}{q(x)} + \frac{r(x)}{q(x)} = \frac{p(x)+r(x)}{q(x)}.$$

Example 3. Find the sum of $\frac{5x+3}{2x+1}$ and $\frac{3x-4}{2x+1}$.

Solution. $\frac{5x+3}{2x+1}+\frac{3x-4}{2x+1}=\frac{5x+3+3x-4}{2x+1}$

$$=\frac{8x-1}{2x+1}.$$

Example 4. Find the sum of $\frac{x-2}{x-1}$ and $\frac{x+1}{x+2}$.

Solution. $\frac{x-2}{x-1}+\frac{x+1}{x+2}=\frac{(x-2)(x+2)+(x+1)(x-1)}{(x-1)(x+2)}$

$$=\frac{x^2-4+x^2-1}{x^2+x-2}$$

$$=\frac{2x^2-5}{x^2+x-2}.$$

EXERCISE 3 (d)

(Section A)

Find the sum of the following pairs of rational expressions :

1. $\frac{x}{x-a}$, $\frac{a}{x-a}$.
2. $\frac{x-3}{x+5}$, $\frac{x-2}{x+5}$.
3. $\frac{x^2+1}{x+3}$, $\frac{x^2-3}{x+3}$.
4. $\frac{x}{x-a}$, $\frac{a}{a-x}$
5. $\frac{x+y}{x}$, $\frac{x-y}{y}$.
6. $\frac{a-b}{ab}$, $\frac{b-c}{bc}$.

(Section B)

Find the sums :

7. $\frac{x+2}{x+3}+\frac{x-1}{x-2}$.
8. $\frac{2}{3x(x+1)}+\frac{1}{9(x+1)}$.
9. $\frac{x}{x^2-36}+\frac{1}{x+6}$.
10. $\frac{x+1}{(x-1)^2}+\frac{1}{x+1}$.
11. $\frac{x^2+xy+y^2}{x+y}+\frac{x^2-xy+y^2}{x-y}$.
12. $\frac{3}{1-25x^2}+\frac{1}{(5x-1)(x+1)}$.

(Section C)

Simplify :

13. $\frac{x^2+x-1}{x^2-1}+\frac{x+1}{x^3+2}$.
14. $\frac{1}{x+y}+\frac{1}{x-y}+\frac{2x}{x^2+y^2}$.
15. $\frac{x+3}{x^2-3x+2}+\frac{x+2}{x^2-4x+3}+\frac{x+1}{x^2-5x+6}$.

3·5. ADDITION PROPERTIES OF RATIONAL EXPRESSIONS

Let us discuss addition properties of rational expressions. It is easy to verify them from what we have discussed so far.

(1) For any two rational expressions $\frac{p(x)}{q(x)}$ and $\frac{r(x)}{s(x)}$,

$$\frac{p(x)}{q(x)}+\frac{r(x)}{s(x)} \text{ is also a rational expression.}$$

We say that the rational expressions are **closed** under addition.

(2) For any two rational expressions $\frac{p(x)}{q(x)}$ and $\frac{r(x)}{s(x)}$,

$$\frac{p(x)}{q(x)}+\frac{r(x)}{s(x)}=\frac{r(x)}{s(x)}+\frac{p(x)}{q(x)}.$$

We say that addition of rational expressions is **commutative.**

(3) For any three rational expressions $\frac{p(x)}{q(x)}$, $\frac{r(x)}{s(x)}$ and $\frac{u(x)}{v(x)}$,

$$\left[\frac{p(x)}{q(x)}+\frac{r(x)}{s(x)}\right]+\frac{u(x)}{v(x)}=\frac{p(x)}{q(x)}+\left[\frac{r(x)}{s(x)}+\frac{u(x)}{v(x)}\right]$$

We say that addition of rational expressions is **associative.**

(4) For any rational expression $\frac{p(x)}{q(x)}$,

$$\frac{p(x)}{q(x)}+\frac{0}{1}=\frac{p(x)}{q(x)}.$$

We say that zero rational expression *i.e.* $\frac{0}{1}$ is the **additive identity** for rational expressions.

(5) For any rational expression $\frac{p(x)}{q(x)}$, there exists another rational expression $-\frac{p(x)}{q(x)}$ such that

$$\frac{p(x)}{q(x)}+\frac{-p(x)}{q(x)}=0.$$

We say that $\frac{-p(x)}{q(x)}$ is the **additive inverse** of $\frac{p(x)}{q(x)}$.

Now we can define subtraction of two rational expressions. If $\frac{p(x)}{q(x)}$ and $\frac{r(x)}{s(x)}$ are rational expressions, we define $\frac{p(x)}{q(x)}-\frac{r(x)}{s(x)}$ as $\frac{p(x)}{q(x)}+\frac{-r(x)}{s(x)}$ which is equal to $\frac{p(x)s(x)-r(x)q(x)}{q(x)s(x)}$.

Example 5. Find the additive inverse of $\frac{x^2-5x}{2x+1}$.

Solution. The additive inverse of $\frac{x^2-5x}{2x+1}$ is $\frac{-(x^2-5x)}{2x+1}$ *i.e.* $\frac{-x^2+5x}{2x+1}$.

Example 6. Express $\frac{2}{(x-1)^2}-\frac{3}{x^2-1}$ as a rational expression.

Solution. $\dfrac{2}{(x-1)^2}-\dfrac{3}{x^2-1}=\dfrac{2}{(x-1)^2}+\dfrac{(-3)}{(x+1)(x-1)}$

$$=\frac{2\,(x+1)+(-3)(x-1)}{(x+1)(x-1)^2}$$

$$=\frac{2x+2-3x+3}{(x+1)(x-1)^2}$$

$$=\frac{-x+5}{(x+1)(x-1)^2}.$$

EXERCISE 3 (e)

(Section A)

1. What is the additive identity for rational expression $\dfrac{2x^2-1}{x+3}$?
2. What is the additive inverse of $\dfrac{x^2+1}{x-1}$?
3. Find the additive inverse of $\dfrac{x^2-3x}{x+2}$.

(Section B)

Express the following as rational expressions :

4. $\dfrac{5}{x-y}-\dfrac{2}{x-y}$
5. $\dfrac{3}{x-y}+\dfrac{1}{y-x}$.
6. $\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}$.
7. $\dfrac{x-5}{x+5}-\dfrac{x+5}{x-5}$.
8. $\dfrac{4x}{x^2-1}-\dfrac{x-3}{x+1}$.
9. $\dfrac{3x}{(x-2)^2}-\dfrac{4}{x-2}$.

(Section C)

10. Simplify : $\dfrac{x^2+2}{x-1}-\dfrac{x-3}{x+1}$.
11. Simplify : $\dfrac{x}{x-y}+\dfrac{y}{x+y}-\dfrac{2xy}{x^2-y^2}$.
12. Simplify : $\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}-\dfrac{3x^2}{x-1}$.
13. Simplify : $\dfrac{4x}{x^2-3x+2}-\dfrac{4}{1-x}-\dfrac{5}{x-2}$.
14. Simplify : $\left(\dfrac{x^2+1}{x-1}+\dfrac{x+1}{2x+1}\right)-\left(\dfrac{x-1}{x+2}+\dfrac{x+1}{x-2}\right)$.

3·6. MULTIPLICATION OF RATIONAL EXPRESSIONS

If $\dfrac{p(x)}{q(x)}$ and $\dfrac{r(x)}{s(x)}$ are two rational expressions, we define their multiplication as,

$$\frac{p(x)}{q(x)}\times\frac{r(x)}{s(x)}=\frac{p(x)\,r(x)}{q(x)\,s(x)}.$$

Example 7. Find the product of $\frac{5x+2}{5x-3}$ and $\frac{x+6}{x+2}$.

Solution. $$\frac{5x+2}{5x-3}\times\frac{x+6}{x+2}=\frac{(5x+2)(x+6)}{(5x-3)(x+2)}$$
$$=\frac{5x^2+32x+12}{5x^2+7x-6}.$$

Example 8. Multiply $\frac{x^2+10x+25}{x^2+5x+6}$ by $\frac{x-3}{x^2-25}$.

Solution. $$\frac{x^2+10x+25}{x^2+5x+6}\times\frac{x-3}{x^2-25}=\frac{(x+5)^2(x-3)}{(x+2)(x+3)(x+5)(x-5)}$$

The factors of the numerator are $(x+5)$, $(x+5)$, $(x-3)$.

The factors of the denominator are $(x+2)$, $(x+3)$, $(x+5)$, $(x-5)$.

$\therefore$ Their g.c.d.$=(x+5)$

Cancelling the g.c.d., we get the product in lowest terms as $\frac{(x+5)(x-3)}{(x+2)(x+3)(x-5)}$

$$=\frac{x^2+2x-15}{x^3-19x-30}.$$

EXERCISE 3 (f)

(Section A)

Find the product of the following pairs of rational expressions and express the product in its lowest terms :

1. $\frac{2x+2}{x-1}, \frac{x+3}{x-2}$.
2. $\frac{x^2+1}{x-1}, \frac{x+1}{x^2-2}$.
3. $\frac{x^2-4}{x+1}, \frac{2x+2}{x-2}$.
4. $\frac{x^2-9}{x+2}, \frac{x^2-4}{x+3}$.

(Section B)

Multiply the following pairs of rational expressions and express the product in its lowest terms :

5. $\frac{x^2+11x+28}{x^2+11x+30}, \frac{x^2-36}{x^2-49}$.
6. $\frac{4x^2-9}{2x^2+5x+3}, \frac{x(x+1)}{2x-3}$.
7. $\frac{x^2+ax}{ax-bx}, \frac{b^2-a^2}{x^2-a^2}$.
8. $\frac{2x^2-5x-3}{ax^2-a}, \frac{2ax^2+ax-3a}{2x^2-3x-9}$.

(Section C)

9. Simplify : $\frac{1-x}{1+y}\times\frac{1-y^2}{x+x^2}\times\frac{1}{1-x}$.
10. Simplify : $\frac{x^3-y^3}{x^2-y^2}\times\frac{x+y}{(x-y)^2}\times\frac{x-y}{x+xy+y^2}$
11. Simplify : $\frac{2x^2+x-1}{x^2-4x+3}\times\frac{2x^2-5x+3}{6x^2+x-2}\times\frac{3x^2-7x-6}{2x^2-7x+6}$.

3·7. MULTIPLICATION PROPERTIES OF RATIONAL EXPRESSIONS

Let us discuss multiplication properties of rational expressions. We can easily verify them from what we have discussed so far.

(1) For any two rational expressions $\frac{p(x)}{q(x)}$ and $\frac{r(x)}{s(x)}$,

$$\frac{p(x)}{q(x)} \times \frac{r(x)}{s(x)} \text{ is also a rational expression.}$$

We say that rational expressions are **closed** under multiplication.

(2) For any two rational expressions $\frac{p(x)}{q(x)}$ and $\frac{r(x)}{s(x)}$,

$$\frac{p(x)}{q(x)} \times \frac{r(x)}{s(x)} = \frac{r(x)}{s(x)} \times \frac{p(x)}{q(x)}.$$

We say that multiplication of rational expressions is **commutative.**

(3) For any three rational expressions $\frac{p(x)}{q(x)}$, $\frac{r(x)}{s(x)}$ and $\frac{u(x)}{v(x)}$,

$$\left[\frac{p(x)}{q(x)} \times \frac{r(x)}{s(x)}\right] \times \frac{u(x)}{v(x)} = \frac{p(x)}{q(x)} \times \left[\frac{r(x)}{s(x)} \times \frac{u(x)}{v(x)}\right].$$

We say that multiplication of rational expressions is **associative.**

(4) For any rational expression $\frac{p(x)}{q(x)}$,

$$\frac{p(x)}{q(x)} \times \frac{1}{1} = \frac{p(x)}{q(x)}.$$

We say that rational expression $\frac{1}{1}$ *i.e.*, 1 is the **multiplicative identity** for rational expressions.

(5) For any rational expression $\frac{p(x)}{q(x)}$,

$$\frac{p(x)}{q(x)} \times \frac{0}{1} = \frac{p(x) \,.\, 0}{q(x) \,.\, 1} = \frac{0}{q(x)}$$
$$= 0$$

The product of any rational expression with zero rational expression is the zero rational expression.

(6) For any non-zero rational expression $\frac{p(x)}{q(x)}$, there exists another rational expression $\frac{q(x)}{p(x)}$ such that

$$\frac{p(x)}{q(x)} \times \frac{q(x)}{p(x)} = 1.$$

We say that $\frac{q(x)}{p(x)}$ is the **multiplicative inverse** or **reciprocal** of $\frac{p(x)}{q(x)}$.

Observe that the product of a rational expression with its reciprocal in lowest terms is always 1.

Now we can define division of a rational expression by a non-zero rational expression. If $\frac{p(x)}{q(x)}$ and $\frac{r(x)}{s(x)}$ are rational expressions and $\frac{r(x)}{s(x)} \neq 0$, then we define

$$\frac{p(x)}{q(x)} \div \frac{r(x)}{s(x)} \text{ as } \frac{p(x)}{q(x)} \times \frac{s(x)}{r(x)} \quad i.e., \quad \frac{p(x)\, s(x)}{q(x)\, r(x)}.$$

Thus, division of a rational expression by a non-zero rational expression is same as multiplying the dividend by the reciprocal of the divisor.

Example 9. Divide $\frac{x^2-1}{x+2}$ by $\frac{(x+1)^2}{x^2+5x+6}$.

Solution.
$$\frac{x^2-1}{x+2} \div \frac{(x+1)^2}{x^2+5x+6} = \frac{x^2-1}{x+2} \times \frac{x^2+5x+6}{(x+1)^2}$$
$$= \frac{(x+1)(x-1)}{(x+2)} \times \frac{(x+2)(x+3)}{(x+1)^2}$$
$$= \frac{(x+1)(x-1)(x+2)(x+3)}{(x+2)(x+1)^2}$$
$$= \frac{(x-1)(x+3)}{x+1}$$
$$= \frac{x^2+2x-3}{x+1}.$$

EXERCISE 3 (g)

(Section A)

1. What is the multiplicative identity of the rational expression $\frac{p(x)}{q(x)}$?

2. What is the multiplicative inverse of the rational expression $\frac{p(x)}{q(x)}$?

What is the restriction on $p(x)$ so that $\frac{p(x)}{q(x)}$ may have a multiplicative inverse ?

3. Find the reciprocals of the following :

(*a*) $\frac{1}{x^2}$, (*b*) $\frac{x+1}{x-1}$, (*c*) $\frac{x^2+x+1}{x-1}$.

4. Divide $\frac{x+4}{x-1}$ by $\frac{x+2}{x-3}$.

5. Divide $\frac{x^2+1}{x-2}$ by $\frac{x^2+1}{x+2}$.

(Section B)

Express the following as rational expressions in lowest terms :

6. $\frac{x^2-36}{x^2-49} \div \frac{x+6}{x+7}$.

7. $\frac{x^2+xy}{x-y} \div \frac{xy}{x^2-y^2}$.

8. $\frac{4x^2-12x+9}{(2x-3)^2} \div (2x-3)$.

9. $\frac{x^2-8x-9}{x^2-17x+72} \div \frac{x^2-1}{x^2-25}$.

10. $\frac{x^2+8x+12}{x^2-7x+12} \div \frac{x^2+4x-12}{x-4}$.

(Section C)

11. Simplify : $\left(\frac{2x^2+3}{x-1} + \frac{x+3}{x+1}\right) \div \frac{x^2-1}{3x}$.

12. Simplify : $\left(\frac{x^3-8}{x^2+4x+4} \times \frac{x+3}{x^2-4}\right) \div \frac{x^2+2x+4}{x^2+3x+2}$.

3·8. You have learnt properties of rational expressions. You now recall properties of rational numbers. Compare properties of rational numbers with those of rational expressions.

What do you observe ?

We find that algebra of rational expressions is *analogous* to that of rational numbers. So, we say that rational expressions behave like rational numbers.

We have used several identities proved for rational numbers. These identities will also be true for rational expressions. So, we can write special products and factorisation for rational expressions.

If R and S are two rational expressions, then the following results or statements will be very helpful to you in solving problems :

(1) $(R+S)^2=R^2+2RS+S^2$

(2) $(R-S)^2=R^2-2RS+S^2$

(3) $(R+S)(R-S)=R^2-S^2$

(4) $(R+S)^3=R^3+S^3+3RS(R+S)$

(5) $(R-S)^3=R^3-S^3-3RS(R-S)$

(6) $R^3+S^3=(R+S)(R^2-RS+S^2)$

(7) $R^3-S^3=(R-S)(R^2+RS+S^2)$

REVIEW EXERCISE III

(Section A)

1. Fill in the blanks to make the following statements true :
   (*a*) Polynomials behave like.........
   (*b*) Every rational number is also a.........expression.
   (*c*) Rational expressions are.........under addition.
   (*d*) Multiplication of rational expressions is.........and associative.
   (*e*) Addition of rational expressions is.........and commutative.
   (*f*) The product of a non-zero rational expression and its reciprocal is always.........
2. Find the additive inverse of $\frac{5-x}{x^2+2}$.
3. Find the reciprocal of $\frac{x^2+1}{x-1}$.

(Section B)

4. Reduce the rational expression $\frac{(x-3)(x^2-5x+4)}{(x-1)(x^2-2x-3)}$ to its lowest terms.
5. Find the sum of $\frac{x^2+1}{x^2-1}$ and $\frac{x+1}{x+2}$.
6. Which rational expression should be added to $\frac{x^3-1}{x^2+2}$ to get $\frac{2x^3-x^2+3}{x^2+2}$?
7. Find the product of $\frac{x^2-7x+10}{(x-4)^2}$ and $\frac{x^2-7x+12}{x-5}$.
8. Divide $\frac{x^2-1}{x+3}$ by $\frac{x^2+2x+3}{x-7}$.

(Section C)

9. Simplify : $\frac{1}{a-1}-\frac{a}{a^2-1}-\frac{a^2}{a^4-1}-\frac{a^4}{a^8-1}$.
10. Simplify : $\frac{x^3-a^3}{x^2-2bx+b^2}\times\frac{x^2-bx-cx+bc}{ax-x^2}\div\frac{x^2-cx}{x-b}$.
11. Simplify : $\left(\frac{x+3}{x+2}\times\frac{x^2-1}{x+6}\right)-\left(\frac{x^2+7}{2}-\frac{x^2+3}{x}\times\frac{4x}{3}\right)$.

□□

4

MENSURATION
(Plane Figures)

4·1. REVIEW

Mensuration is the science of measurement. It deals with determining lengths, breadths, areas and volumes of plane figures and solids. You have learnt to find area of plane figures in the earlier classes. Let us recall those results.

1. Rectangles.

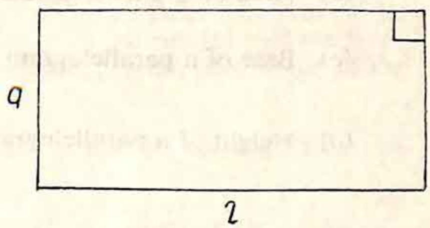
(*a*) Perimeter of a rectangle$=2\,(l+b)$

(*b*) Area of a rectangle, $A=l\times b$

(*c*) Length of a rectangle$=\frac{A}{b}$

(*d*) Breadth of a rectangle$=\frac{A}{l}$

(*e*) Diagonal of a rectangle$=\sqrt{(l^2+b^2)}$



2. Squares.

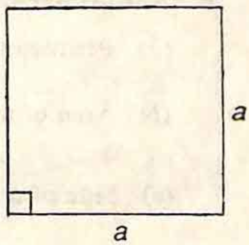
(*a*) Perimeter of a square, $p=4a$

(*b*) Side of a square $=\frac{p}{4}$

(*c*) Area of a square, $A=a^2$

(*d*) Side of a square $=\sqrt{A}$

(*e*) Diagonal of a square $=a\sqrt{2}$



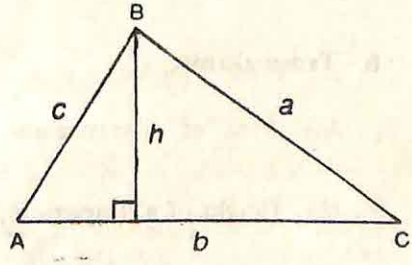
3. Triangles.

(*a*) Semi-perimeter of a triangle,

$$s=\frac{a+b+c}{2}$$

(*b*) Area of a triangle$=\frac{1}{2}\,bh$

(*c*) Area of a triangle$=\sqrt{s(s-a)(s-b)(s-c)}$

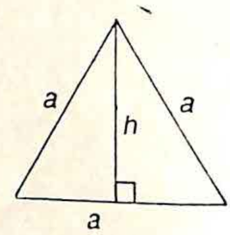


(*d*) Area of an equilateral triangle

$$=\frac{\sqrt{3}}{4}a^2$$

Altitude of an equilateral triangle,

$$h=\frac{\sqrt{3}}{2}a$$

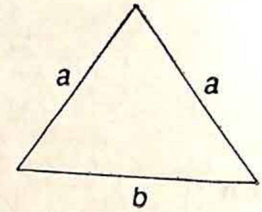


(*e*) Perimeter of an isosceles triangle,

$$p=a+b+a$$

Area of an isosceles triangle,

$$=\frac{a}{4}\sqrt{4b^2-a^2}$$



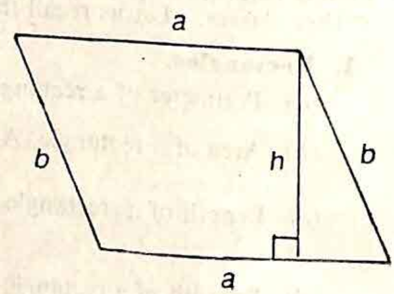
4. Parallelograms.

(*a*) Perimeter of a parallelogram, $p=2(a+b)$

(*b*) Area of a parallelogram, $A=ah$

(*c*) Base of a parallelogram $=\frac{A}{h}$

(*d*) Height of a parallelogram $=\frac{A}{a}$

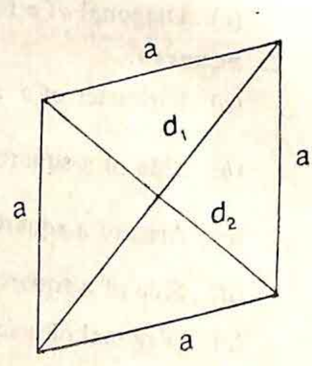


5. Rhombuses.

(*a*) Perimeter of a rhombus, $p=4a$

(*b*) Area of a rhombus, $A=\frac{1}{2}d_1d_2$

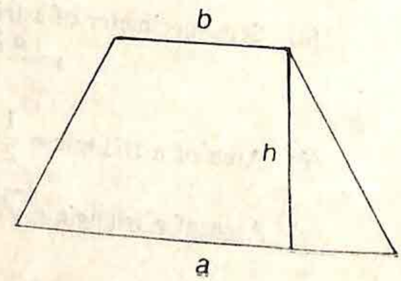
(*c*) Side of a rhombus, $a=\frac{1}{2}\sqrt{(d_1^2+d_2^2)}$



6. Trapeziums.

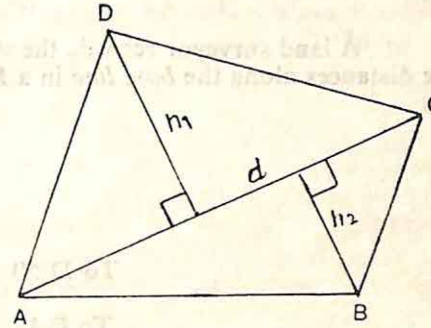
(*a*) Area of a trapezium, $A=\frac{1}{2}(a+b)h$

(*b*) Height of a trapezium, $h=\frac{2A}{a+b}$



7. Quadrilaterals.

Area of a quadrilateral, $A = \frac{1}{2}(h_1 + h_2)\,d$



EXERCISE 4 (a)

1. Find the least number of square slabs of equal size that will cover the floor of a hall which is 18 metres long and 14 metres broad.

2. If the length of a rectangle is decreased by 5 metres, and its breadth increased by two metres, it becomes a square whose area is 65 sq. metres less than that of the original rectangle. Find the dimensions of the rectangle.

3. A garden consists of a rectangular plot of grass surrounded by a border 3 metres wide, the whole being 40 metres long and 24 metres wide. In the central part are three rectangular flower-beds each 4 metres long and 3 metres wide. What percentage of the whole area is grass-covered ?

4. The sides of a triangular field are 1,022 metres, 1,095 metres and 949 metres. It is let out at Rs. 100 per hectare. Find the rent.

5. A plot of ground is in the form of an isosceles triangle. If it costs Rs. 1,000 at the rate of Rs. 2·50 per square metre, and if each of the equal sides measures 40 metres, find the length of the base.

6. Find the base of an isosceles triangle whose area is 60 cm^2 and the length of one of its equal sides is 13 *cm*.

7. The perimeter of a rhombus is 146 *cm* and one of its diagonals is 55 *cm*. Find the other diagonal and area of the rhombus.

8. A trapezium with parallel sides of length as 7 : 3 is cut from a rectangle 30 *dm* by 4 *dm* so as to have an area of one-third the latter. Find the lengths of the parallel sides.

9. The length and breadth of a room are in the ratio of 3 : 2. Its height is equal to $\frac{1}{2}$ of its length. The cost of carpeting the floor at Rs. 4 per sq. metre is Rs. 216. Find the cost of papering the walls at Rs. 3·20 per square metre.

10. The perimeter of a right triangle is p. Its hypotenuse is d. Find the other two sides and the area of the triangle. Taking $p = 12$ *cm* and $d = 5$ *cm*, solve the problem.

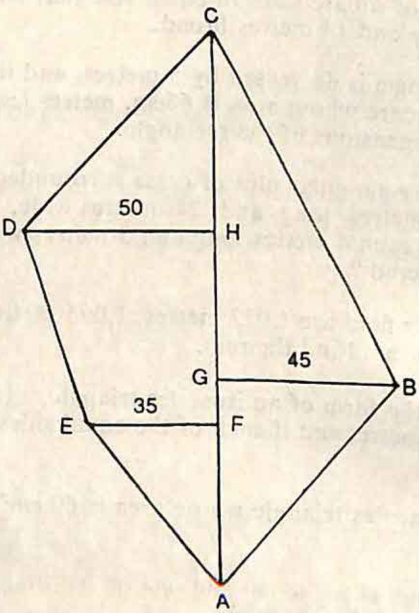
4·2. AREAS OF IRREGULAR FIGURES

When a Tehsildar goes to check measurements of land in a village, he does not often come across fields whose shapes are rectangles, squares or right-angled triangles. He has to measure *irregular polygons*. To find the area of such polygons, we divide the figure into small rectangles, trapeziums and right-angled triangles. We, then, can find the area of each of the regular figures so formed. Adding all these areas, we get the area of the irregular polygon.

A land surveyor records the various measurements such as lengths of the offsets and the distances along the *base line* in a **field book,** in the following form :

	Metres	
	To C	
	140	
To D 50	90	
	50	45 To B
To E 35	40	
	From A	

Let us first draw the figure. Note that the field book is read from the *bottom upwards.*



Taking a suitable scale, 20 $m=1$ cm, we draw base line $AC=140\ m$. Measure $AF=40$ m on AC and draw offsect $FE=35\ m$ to the left.

Measure $AG=50\ m$ on AC and draw offset $GB=45\ m$ to the right.

Measure $AH=90\ m$ on AC and draw offset $HD=50\ m$ to the left.

Then by joining A, B, C, D, E and A, we get the polygon $ABCDE$.

Area of the field $ABCDE$

$=$Area of $\triangle ABG+$Area of $\triangle GBC+$Area of $\triangle HCD+$Area of trap. $FHDE$ $+$Area of $\triangle AFE$.

Area of right-angled $\triangle ABG=\frac{1}{2}\times AG\times GB$

$=\frac{1}{2}\times 50\times 45\ sq.\ m$

$=1125\ m^2$

Area of right-angled $\triangle GBC=\frac{1}{2}\times GC\times GB$ $\quad [GC=AC-AG=(140-50)\ m=90\ m]$

$=\frac{1}{2}\times 90\times 45\ sq.\ m$

$=2025\ m^2$

Area of right-angled $\triangle HCD=\frac{1}{2}\times HC\times HD$ $\quad [HC=AC-AH=(140-90)\ m=50\ m]$

$=\frac{1}{2}\times 50\times 50\ sq.\ m$

$=1250\ m^2$

Area of trap. $FHDE$ $=\frac{1}{2}(FE+HD)\times FH$ $[FH=AH-AF=(90-40)\ m=50\ m]$

$=\frac{1}{2}(35+50)\times 50\ sq.\ m$

$=2125\ m^2$

Area of right-angled $\triangle AFE=\frac{1}{2}\times AF\times FE$

$=\frac{1}{2}\times 40\times 35\ sq.\ m$

$=700\ m^2$

$\therefore$ Area of pentagonal field $ABCDE$

$=1125\ m^2+2025\ m^2+1250\ m^2+2125\ m^2$

$=7225\ m^2$.

EXERCISE 4 (b)

(Section A)

1. Find the area of a pentagonal field, given the following data in a land surveyor's field book :

	Metres	
	To C	
	150	
To D 20	120	
	80	To B 50
To E 30	50	
	From A	

2. Find the area of a pentagonal field, given the following measurements in a land surveyor's field book :

	Metres	
	To D	
	130	
	80	To C 50
To E 60	40	
	25	To B 25
	From A	

(Section B)

From the given measurements of the fields, draw figures of appropriate scale and find their areas :

3.

	Metres	
	100	
	80	To C 40
To E 25	70	
	40	To B 30
To F 20	30	
	From A	

4.

	Metres	
	To D	
	550	
To E 60	410	
	320	To C 50
To F 20	200	
	110	To B 10
	From A	

(Section C)

The measurements of fields are noted as follows by a surveyor in his field book. Draw the plan of the fields according to suitable scale and find their areas :

5.

	Metres	
	To E	
	1000	
To F 230	840	
	700	To D 60
	610	To C 800
To G 340	560	
	310	To B 220
	From A	

6.

	Metres	
	To A	
	600	
To G 140	540	
To F 150	480	
	470	To B 100
To E 50	380	
	100	To C 150
	From D	

4·3. CIRCLES

You know how to draw a circle when its centre and radius are specified. You also know that the perimeter of a circle is called its **circumference**.

If A and C denote the area and circumference respectively of a circle of radius r, then

$$A = \pi r^2 \quad ; \quad C = 2\pi r.$$

You know that the circumference of a circle bears a *constant ratio* to its diameter. This constant ratio is denoted by the Greek letter π, pronounced '*pi*'. It stands for an *irrational* number whose value is given approximately to two decimal places by 3·14 or by the fraction $\frac{22}{7}$. Upto four decimal places, its approximate value is 3·1416. But for calculation purposes an approximate value of π is taken as $\frac{22}{7}$ or 3·14.

EXERCISE 4 (c)

1. A wire is in the form of a circle of radius 42 *cm*. Determine the sides of the square into which it can be bent. $\left(\text{Use } \pi = \frac{22}{7}\right)$ [*C.B.S.E., 1977 (Delhi)*]

2. What is the area of the circle, the circumference of which is equal to the perimeter of a square of side 11 cm ?

3. From a copper plate which is a square of side 12·5 *cm*, circular disc of diameter 7 *cm* is cut off. Find the weight of the remaining part, if 1 *sq. cm* of the plate weighs 0·8 gram. $\left(\text{Assume } \pi = \frac{22}{7}\right)$ [*C.B.S.E., 1980 (Delhi)*]

4. The outer circumference of a circular race track is 528 *m*. The track is everywhere 14 m wide. Calculate the cost of levelling the track at the rate of 50 paise per *sq. m*. $\left(\text{Use } \pi = \frac{22}{7}\right)$. [*C.B.S.E., 1982 (Delhi)*]

5. The inner circumference of a circular track is 440 *m*. The track is 14 *m* wide. Calculate the cost of (*a*) levelling the track at the rate of 20 paise per *sq. m* (*b*) putting up fence along outer circle at Rs. 2 per *m*. $\left(\text{Use } \pi = \frac{22}{7}\right)$

6. A playground has the shape of a rectangle, with two semi-circles on its smaller sides as diameters, added to its outside. If the sides of the rectangle are 36 *m* and 24·5 *m*, find the area of the playground.

7. A piece of wire that has been bent into the form of a semi-circle, including the bounding diameter, is straightened and then bent into the form of a square. The diameter of the circle is 12 *cm* long. Which area is larger, the semi-circle or the square ? By how much ?

8. The cost of turfing a uniform circular road round a circular garden at 20 P per sq. metre is Rs. 215·60 and the area of the garden is 1,386 sq. metres. Find the breadth of the circular road.

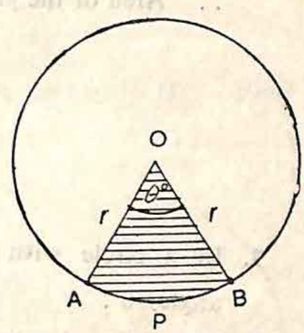
9. Four circular coins, each of radius 1·4 *cm*, are placed flat on a table such that their centres are the corners of a square and that each coin touches two of the others. Calculate the area lying vacant between their rims.

10. A boy is cycling such that the wheels of the cycle are making 140 revolutions per minute. If the diameter of the wheel is 60 *cm*, calculate the speed per hour with which the boy is cycling.

11. A circular field has a perimeter of 660 *m*. A plot in the shape of a square having its vertices on the circumference of the field is marked in the field. Calculate the area of the square plot.

4·4. SECTOR

The shaded region shown on the right is a part of the circular region with centre O and radius r. The shaded region $OAPB$ is called a **sector** of the circle. Its boundary consists of arc APB and two radii OA and OB. This sector has an angle θ, subtended at the centre of the circle.



The region bounded by two radii of a circle and the arc intercepted by them is called the sector of the circle.

When $\theta < 180°$, arc AB is a minor arc. When θ is increased to 180°, arc AB increases proportionally, to the size of a semi-circular arc of length πr. So, half the circumference of a circle *i.e.* πr subtends an angle of 180° at the centre of the circle.

Thus, for sector AOB with $\angle AOB = \theta°$, the length of minor arc AB is $\frac{\theta}{180} \times \pi r$ *i.e,* $\frac{\pi r\theta}{180}$

$$\therefore \qquad \mathbf{l = \frac{\pi r \theta}{180}}.$$

Observe that the area of a sector is proportional to the angle of the sector. When the arc subtends an angle of 180° at the centre, the area of the corresponding sector is $\frac{1}{2}$ (area of the circle) *i.e.* $\frac{1}{2}\pi r^2$.

Thus, for sector AOB with $\angle AOB = \theta°$, the area of the sector is $\frac{\theta}{180} \times \frac{1}{2}\pi r^2$ *i.e.,* $\frac{\pi r^2 \theta}{360}$

$$\therefore \qquad A = \frac{\pi \mathbf{r}^2 \theta}{\mathbf{360}}.$$

Area can also be expressed in terms of l.

$$A = \frac{\pi r^2 \theta}{360} \qquad = \frac{\pi r \theta}{180} \times \frac{r}{2} \qquad = l \times \frac{r}{2}$$

$$\therefore \qquad A = \frac{\mathbf{1}}{\mathbf{2}}\,\mathbf{lr}$$

Example 1. In a circle of radius $3{\cdot}5$ *cm*, find the perimeter of a sector with central angle $36°$.

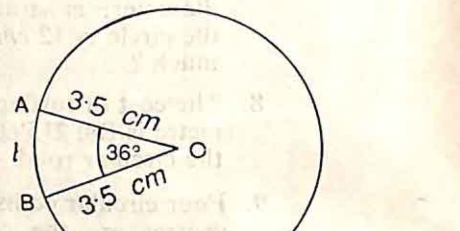
Solution. Here $\theta=36°$, $r=3{\cdot}5\ cm$

Length of the arc, $l=\dfrac{\pi r\theta}{180}$

$$=\frac{\frac{22}{7}\times 3{\cdot}5\times 36}{180}\ cm$$

$$=\frac{22\times 3{\cdot}5\times 36}{7\times 180}\ cm$$

$$=2{\cdot}2\ cm$$



$\therefore$ Perimeter of the sector$=r+l+r$

$$=(3{\cdot}5+2{\cdot}2+3{\cdot}5)\ cm$$

$$=9{\cdot}2\ cm.$$

Example 2. Find the area of a sector of a circle whose radius is 6 *dm* and central angle is $105°$.

Solution. Here $\theta=105°$, $r=6\ dm$

$$\text{Area of a sector}=\frac{\pi r^2\theta}{360}$$

$\therefore$ Area of the given sector$=\dfrac{22}{7}\times\dfrac{6\times 6\times 105}{360}\ dm^2$

$$=\frac{22\times 105}{7\times 10}\ dm^2$$

$$=33\ dm^2$$

EXERCISE 4 (d)

(Section A)

1. In a circle with radius 21 *cm*. Find the length of the arc of a sector with central angle $60°$.
2. In a circle of radius $3{\cdot}5$ *cm*, find the area of the sector whose angle measures $36°$. $\left(\text{Take } \pi=\dfrac{22}{7}\right)$
3. Find the area of a sector of a circle whose radius is 8 *cm* and the length of the arc is 15 *cm*.
4. Find the area of a sector of a circle whose radius is 6 metres and central angle is $42°$.

(Section B)

5. The length of the minute hand of a wall clock is $10{\cdot}5$ *om*. Find the area swept by the minute hand in 10 minutes time. $\left(\text{Use } \pi=\dfrac{22}{7}\right)$
6. Given a circle with radius $3{\cdot}6$ *cm*. Find the perimeter and area of its sector with central angle $36°$. $\left(\text{Use } \pi=\dfrac{22}{7}\right)$
7. The area of a sector is $\dfrac{1}{10}$ that of the whole circle. Find the angle of the sector.
8. From a circular piece of card-board of radius 3 *cm* two sectors with central angle of $40°$ each have been cut off. Find the area of the remaining portion.

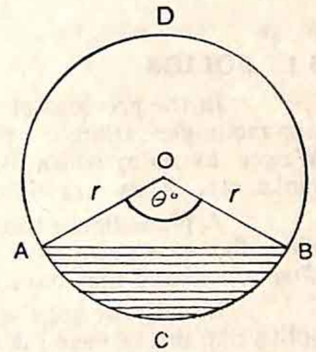
(Section C)

9. A circular disc of 4 cm in radius is divided into three sectors with central angles 110°, 150° and 100°. What part of the whole disc is the sector with the central angle 150° ?

10. A circular disc of 6 cm in radius is divided into three sectors with central angles 120°, 150°, 90°. What part of the whole area is the sector with central angle 120° ? Also give the ratio of the areas of the sectors. [*C.B.S.E., 1977 (A.I)*]

4·5. SEGMENT OF A CIRCLE

The drawing on the right shows a circle with centre O and radius r.



Let chord AB divide the circle into two segments ACB and ADB.

The segment ACB which is *less* than the semi-circle, is called the **minor segment**. The segment ADB which is *greater* than the semi-circle, is called the **major segment**.

Join OA and OB. Let $\angle AOB$ be $\theta°$.

Let us now find the area of the minor segment shown by shaded region.

Area of the minor segment ACB

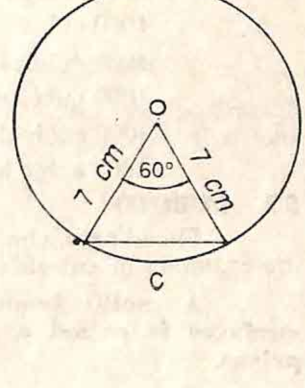
$=$Area of sector $OACB-$Area of $\triangle AOB$.

Then, area of the major segment ADB

$=$Area of the circle$-$Area of minor segment ACB.

Example 3. The radius of a circle is 7 cm and the angle of the sector is 60°. Find the area of the minor segment.

Solution. Here $r=7$ cm, $\theta=60°$



Area of sector $OACB=\dfrac{\pi r^2\theta}{360}$

$=\dfrac{22\times7\times7\times60}{7\times360}$ cm²

$=\dfrac{77}{3}$ cm² $=25{\cdot}667$ cm²

Area of $\triangle AOB$ $=\dfrac{\sqrt{3}}{4}a^2$ [*$\triangle AOB$ is equilateral*]

$=\dfrac{\sqrt{3}\times7\times7}{4}$ cm²

$=\dfrac{49\times1{\cdot}732}{4}$ cm²

$=21{\cdot}217$ cm²

$\therefore$ Area of the minor segment

$=$Area of the sector $OACB-$Area of the $\triangle AOB$

$=25{\cdot}667$ cm²$-21{\cdot}217$ cm² $=4{\cdot}45$ cm².

EXERCISE 4 (e)

1. A chord AB of a circle of radius 15 cm makes an angle of 60° at the centre of the circle. Find the area of the minor segment. (Take $\pi=3{\cdot}14$, $\sqrt{3}=1{\cdot}73$).
2. The radius of a circle with centre O is 5 cm. Two radii OA and OB are drawn at right angles to each other. Find the area of the two segments made by the chord BA. (Take $\pi=3{\cdot}14$).
3. Find the area of the minor segment of a circle, given that the angle of the sector is 120° and the radius of the circle is 21 cm. (Take $\pi=3{\cdot}1416$).
4. A chord AB of a circle of radius 10 cm makes a right angle at the centre of the circle. Find the area of the major and the minor segments. (Take $\pi=3{\cdot}14$).

□ □

5

MENSURATION—SOLIDS

5·1. SOLIDS

In the previous chapter you learnt about methods for finding areas of plane figures like rectangles, triangles, polygons, circles, sectors and segments. We shall now discuss solids. We see everyday solids like a brick, a book, a pencil, a match-box, a tennis ball, sugar cubes, tanks, etc. These are different kinds of solids.

A plane figure lies entirely in a plane where as a solid does not. Solids lie in space. Plane figures are *two dimensional* and they have length and breadth only. But solids are *three dimensional* and they have length, breadth and height or thickness.

Usually we see the outside of a solid ; all the outside is called the **surface**. Some solids like bricks have plane surfaces, other like tennis balls have curved surfaces.

The amount of space enclosed by the bounding surface or surfaces of a solid is called the volume of the solid.

As with area we need a standard unit of volume.

The unit of measurement of volume is **unit cube**. It is a cube with 1 cm sides. Its volume is 1 cubic centimetre or 1 cm^3.

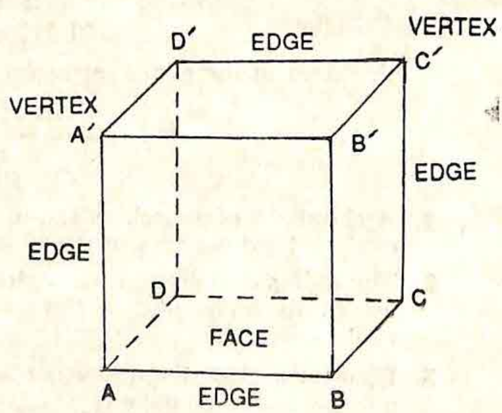
We give below various units of volume depending upon the unit of measure.

1000 cubic millimetres	= 1 cubic centimetre	(cubic cm)
1000 cubic centimetres	= 1 cubic decimetre	(cubic dm)
1000 cubic decimetres	= 1 cubic metre	(cubic m)
1000 cubic metres	= 1 cubic decametre	(cubic dam)
1000 cubic decametres	= 1 cubic hectometre	(cubic hm)
1000 cubic hectometres	= 1 cubic kilometre	(cubic km)

5·2. CUBOID

Shoe-boxes, chalk-boxes, match-boxes, packets of surf washing powder, water tub, etc., are examples of **cuboids.**

A solid bounded by six rectangular surfaces is called a cuboid or a rectangular prism.



A cuboid is also called a **rectangular parallelopiped.**

The dimensions of a cuboid are length, breadth and height.

A cuboid has six faces, each one being a rectangle. Opposite faces are parallel and congruent. There are three pairs of parallel faces. Two adjacent faces meet along a line segment called an **edge.**

A cuboid has twelve edges. Opposite edges are parallel and equal in length. There are six pairs of parallel edges. Three mutually perpendicular edges meet at a point called a **vertex,**

A cuboid has eight vertices.

Thus, a cuboid has six faces, twelve edges and eight vertices.

Note that three faces and three edges meet at each vertex.

In the adjoining figure, the faces are rectangles $ABCD$, $A'B'C'D'$, $AA'B'B$, $BB'C'C$, $CC'D'D$ and $DD'A'A$. The sum of the areas of these six faces is the total surface area of the cuboid.

Its edges are AA', BB', CC', DD', AB, $A'B'$, BC, $B'C'$ CD, $C'D'$, AD and AD'. It is easily seen that

$$AB=A'B'=CD=C'D'=l \text{ (length)}$$
$$BC=B'C'=AD=A'D'=b \text{ (breadth)}$$
and
$$AA'=BB'=CC'=DD'=h \text{ (height)}$$

Its vertices are A, B, C, D, A', B', C' and D'.

Volume of a cuboid

$$=\text{length}\times\text{breadth}\times\text{height}$$

Using l, b and h for length, breadth and height of a cuboid and V for its volume, we have

$$\mathbf{V=l\times b\times h}$$

From this formula, we get the following :

(*i*) $l=\dfrac{V}{b\times h}$ (*ii*) $b=\dfrac{V}{l\times h}$ (*iii*) $h=\dfrac{V}{l\times b}$

The total surface area of a cuboid is the sum of the areas of its six faces.

Total surface area of a cuboid $=2(lb+bh+lh)$

Diagonal of a cuboid $=\sqrt{(l^2+b^2+h^2)}$

Note that these formulae are true also when l, b and h are any real numbers.

Example 1. A rectangular tank measuring internally 37 m in length, 12 m in breadth and 8 m in depth, is full of water. Find the weight of water in metric tonnes, given that one cubic metre of water weighs 1,000 kg.

Solution. Length of the tank$=37$ m

Breadth of the tank$=12$ m

Depth of the tank$=8$ m

$\therefore$ Volume of the tank$=37\times12\times8$ cubic metres

$\therefore$ Volume of water in tank$=37\times12\times8$ cubic m

$=3552$ cubic m

Weight of 1 cubic metre of water$=1000$ kg

Weight of water in the tank$=3552\times1000$ kg

$=3552000$ kg

$=$**3552 metric tonnes.**

Example 2. A box with a lid is made of planking 2·5 cm thick. If its external dimensions be 1 m, 85 cm and 65 cm, how many square metres of planking are used in the construction ?

Solution. External dimensions of the box :

length$=1$ m$=100$ cm ; breadth$=85$ cm ; height$=65$ cm

$\therefore$ External volume of the box$=100\times85\times65$ cm^3

$=5,52,500$ cm^3

Internal dimensions of the box :

Length $=100$ cm $-2\times2{\cdot}5$ cm $=(100-5)$ cm $=95$ cm

Breadth $=85$ cm $-2\times2{\cdot}5$ cm $=(85-5)$ cm $=80$ cm

Height $=65$ cm $-2\times2{\cdot}5$ cm $=(65-5)$ cm $=60$ cm

$\therefore$ Internal volume of the box $=95\times80\times60$ cm^3

$=4{,}56{,}000$ cm^3

$\therefore$ Volume of the wood $=5{,}52{,}500$ cm$^3-4{,}56{,}000$ cm^3

$=96{,}500$ cm^3 $\quad={\cdot}0965$ m^3

Thickness of wood plank $=2{\cdot}5$ cm $={\cdot}025$ m

Then area of planking $=\dfrac{{\cdot}0965}{{\cdot}025}$ m^2 $\quad=\dfrac{965}{250}$ m^2

$=\mathbf{3{\cdot}86\ m^2}$.

EXERCISE 5 (a)

(Section A)

1. Find the volume and surface of a cuboid whose dimensions are 36 m, 12 m and 1 m.
2. Find in litres of cubic contents of a tank 2·8 metres long, 1·4 metres wide, and 0·75 metre deep.
3. The area of a playground is 4800 sq. m. Find the cost of covering it with gravel 1 cm deep, if the gravel costs Rs. 4·80 per cubic metre.
4. The outer measurements of a closed wooden box are 42 cm, 30 cm and 27 cm. If the box is made of wood, 1 cm thick, determine the capacity of the box.
5. The length and breadth of a rectangular solid are respectively 25 cm and 20 cm. If the volume is 7000 cm^3, find its height.
6. A brick measures 20 cm by 10 cm by 7·5 cm. How many bricks will be required for a wall 25 cm long, 2 m high and 75 cm thick ?
7. If 72 cubic metres of sand be thrown into a tank 12 metres long and 5 metres wide, find how much the water will rise.

(Section B)

8. A tank 8 metres long, 5 metres wide, contains 800 quintals of water. Find its depth if 1 cubic metre of water weighs 10 quintals.
9. A closed tea chest is 47 cm long, 47 cm wide and 60 cm deep, by internal measurement. Find in square metres the total area of tinfoil needed for lining it.
10. The areas of three adjacent faces of a cuboid are x, y and z. If the volume is V, prove that $V^2=xyz$. [*C.B.S.E., 1982(A.I.)*]
11. A rectangular box whose external dimensions including the lid are 32, 27, 12 decimetres is made of wood ·5 dm in thickness. What is the volume of wood in it ?
12. The annual rainfall at a place is 43 cm. Find the weight in metric tonnes of the annual rain falling there on a hectare of land, taking the weight of water to be 1 metric tonne to the cubic metre.
13. A cubic metre of gold is extended by hammering so as to cover an area of 2 hectares. Find the thickness of the gold in decimals of a cm, correct to the first two significant figures.
14. A rectangular tank is 3·5 m long, 1·8 m broad and 1·5 m deep. Calculate in litres the amount of water it will hold. If 700 litres of water are drawn off, find to the nearest mm, how much the water level sinks ?
15. A stream which flows at a uniform rate of 2 5 km an hour, is 20 metres wide, the depth of a certain ferry being 1·2 metres. How many litres pass the ferry in a minute ?

16. The length, breadth and height of a rectangular solid are in the ratio of 5 : 4 : 2. If the total surface area is 1216 cm², find the length, breadth and height of the solid.

17. The volume of a rectangular solid is 576 cubic cm and the length of a diagonal is $\sqrt{244}$ cm. If its thickness be 6 cm, find its length and breadth.

(Section C)

18. A field is 30 m long and 18 m broad. A pit, 6 m long, 4 m wide and 3 m deep, is dug out from the middle of the field and the earth removed is evenly spread over the remaining area of the field. Find the rise in the level of the remaining part of the field in centimetres correct to one decimal place.

19. What depth of trench 9 dm wide must be dug round a plot 180 dm wide and 240 dm long in order that the earth removed may be sufficient to raise the level of the whole plot by 3 cm ?

20. A closed cistern, 75 cm long, 28 cm wide (external measurements), is made of metal 15 mm thick and has a capacity of 27 litres. Find the external height.

5·3. CUBES

When the dimensions of a retangular solid are equal to one another, it is called a **cube.** In case of a cube, $l=b=h$

Then $V=l\times l\times l \quad =l^3$

The volume of a cube with a side of l units is l^3.

In symbols, $\quad V=l^3$

The edge of a cube, $\quad l=\sqrt[3]{V}$

Total surface area of a cube $\quad =2(l\times l+l\times l+l\times l)$

$=2(l^2+l^2+l^2)$

$=6l^2$

Example 3. Three cubes of metal whose edges are 3 cm, 4 cm, and 5 cm respectively, are melted down and formed into a single cube.. Find the edge of the new cube.

Solution. Edge of the first cube $=3$ cm

$\therefore$ Volume of the first cube $=3^3 \text{ cm}^3 \quad =27 \text{ cm}^3$

Edge of the second cube $=4$ cm

$\therefore$ Volume of the second cube $=4^3 \text{ cm}^3 \quad =64 \text{ cm}^3$

Edge of the third cube $=5$ cm

$\therefore$ Volume of the third cube $=5^3 \text{ cm}^3 \quad =125 \text{ cm}^3$

Then volume of the new cube $=(27+64+125) \text{ cm}^3$

$=216 \text{ cm}^3$

Therefore, edge of the new cube $=\sqrt[3]{216}$ cm

$=\sqrt[3]{6\times 6\times 6}$ cm $\quad =$ **6 cm.**

EXERCISE 5 (b)

(Section A)

1. Find the volume and surface of a cube whose edge is 15 cm ?
2. The length of the edge of a cube is 4 cm. Find (*i*) the total surface area of the cube, (*ii*) the volume of the cube.
3. The perimeter of one face of a cube is 20 cm. Find (*i*) the total area of six faces, (*ii*) the volume of the cube.
4. What number of 4 cm cubes can be cut from a 12 cm cube ?

5. Two cubes each of side 12 cm are joined end to end. Find the surface area of the resulting cuboid.

6. The surface of a cube is 726 dm^2. Find its volume.

7. A cubical block of stone contains 5,832 cubic cm. Find the length of its side.

(**Section B**)

8. Find the edge of a cube whose surface has the same area as that of a rectangular solid which is 10 dm long, 7 dm broad and 6 dm thick.

9. The three co-terminus edges of a rectangular solid are 36, 75 and 80 cm respectively. Find the edge of a cube which will be of the same capacity.

10. The internal measurements of a box are 20 cm long, 16 cm wide and 24 cm high. How many 4 cm cubes could be put into the box ?

11. Three cubes whose edges are 6 cm, 8 cm and 10 cm respectively are melted without any loss of metal into a single cube. Find the surface area of the new cube.

12. Two cubes, each with 12 cm edge, are joined end to end. Find the surface area of the resulting cuboid. [*C.B.S.E., 1978* (*Delhi*)]

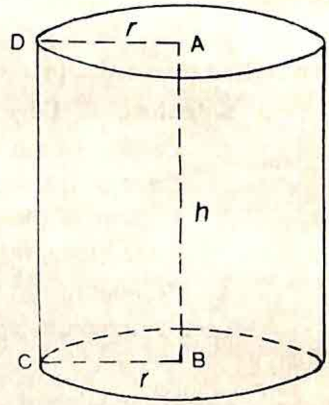
(**Section C**)

13. A cube whose edge is 20 cm long has a circle 20 cm diameter on each of its faces painted black. What is the total area of the unpainted surface of the cube.

14. A rectangular container whose base is a square of side 6 cm, stands on a horizontal table and holds water upto 1 cm from the top. When a cube is placed in the water and is completely submerged, the water rises to the top and 2 cm^3 of water overflows. Calculate the volume of the cube and the length of its edge.

5·4. CYLINDER

Observe a ruler, an ordinary lead pencil, a stone road-roller, a water pipe or a circular pillar. Each of these objects has a shape of a **cylinder**.



A cylinder has a curved surface, two equal circular ends and two equal edges.

The line segment joining the centres of the two circular ends is called the **axis** of the cylinder.

If the axis of the cylinder is perpendicular to the circular ends, then the cylinder is called a **right circular cylinder**.

In the figure, AB is the axis and BC or AD is the radius of the cylinder. The length of the axis is called the **height** of the cylinder.

A right circular cylinder is generally defined as the solid generated by the revolution of a rectangle about one of its sides.

The length of the side AB about which the rectangle $ABCD$ is rotated, is the **height** of the cylinder. The length of the other side BC is the **radius** of the cylinder.

We state the following formulae for a cylinder with radius r and height h :

Volume of a right circular cylinder $=\pi r^2 h$

Curved surface of a right circular cylinder $=2\pi rh$

Whole surface of a right circular cylinder $=$ curved surface $+2\times$(area of the end)

$$=2\pi rh+2\pi r^2 \;=2\pi r(h+r)$$

Observe that the cross-section of a right circular cylinder by a plane parallel to the base is a *circle*.

Example 4. Find the volume of a right cylinder which has a height of 14 dm and a base of radius 3 dm. Also find the curved surface of the cylinder.

Solution. Radius of the base $(r)=3$ dm

Height of the cylinder $(h)=14$ dm

Volume of a right circular cylinder $=\pi r^2 h$

$\therefore$ Volume of the given cylinder $=\frac{22}{7}\times 3\times 3\times 14$ dm^3

$=$**396 cubic dm**

Curved surface of a right circular cylinder

$=2\pi rh$

$\therefore$ Curved surface of the given cylinder $=2\times\frac{22}{7}\times 3\times 14$ sq. dm

$=$**264 sq. dm.**

Example 5. Find the weight of a lead pipe 3·5 metres long. The external diameter of the pipe is 2·4 cm and the thickness of the lead is 2 mm and 1 cm^3 of lead weighs 11·4 grams.

Solution. External radius of the pipe$=\frac{2{\cdot}4}{2}$ cm$=1{\cdot}2$ cm

Length of the pipe $(h)=3{\cdot}5$ m $=350$ cm

Volume of a right circular cylinder $=\pi r^2 h$

$\therefore$ External volume of the pipe $=\frac{22}{7}\times 1{\cdot}2\times 1{\cdot}2\times 350$ cubic cm

$=1584$ cm^3

Thickness of the pipe$=2$ mm $=0{\cdot}2$ m

Internal radius of the pipe $=1{\cdot}2$ cm$-0{\cdot}2$ cm$=1$ cm

$\therefore$ Internal volume of the pipe $=\pi R^2 h$

$=\frac{22}{7}\times 1\times 1\times 350$ cm^3

$=1100$ cm^3

Volume of lead used in the pipe $=(1584-1100)$ cm^3 $=484$ cm^3

Weight of 1 cm^3 of lead $=11{\cdot}4$ grams

$\therefore$ Weight of the lead pipe $=484\times 11{\cdot}4$ grams

$=$**5·5176 kg.**

Example 6. Water is flowing at the rate of 3 km an hour through a circular pipe of 20 cm internal diameter into a circular cistern of diameter 10 m and depth 2 m. In how much time will the cistern be filled ?

Solution. Radius of the cistern$=\frac{10}{2}$ m$=5$ m

Depth of the cistern$=2$ m

Volume of a right cylinder$=\pi r^2 h$

$\therefore$ Volume of the circular cistern$=\frac{22}{7}\times 5^2\times 2$ m^3

$=\frac{1100}{7}$ m^3

Internal radius of the circular pipe $=\frac{20}{2}$ cm

$$=10 \text{ cm} \qquad = \frac{1}{10} \text{ m}$$

Water is flowing through a circular pipe in the form of a right circular cylinder.

Length of this circular cylinder (in one hour) $=3$ km

$$=3000 \text{ m}$$

Volume of water poured into the cistern in one hour $=\pi r^2 l$

$$=\frac{22}{7}\times\frac{1}{10}\times\frac{1}{10}\times 3000 \text{ m}^3$$

$$=\frac{660}{7} \text{ m}^3$$

Time taken to fill the cistern $=\frac{1100}{7}\div\frac{660}{7}$ hours

$$=\frac{1100}{7}\times\frac{7}{660} \text{ hours} \qquad =\frac{5}{3} \text{ hours}$$

$=$ **1 hour 40 minutes.**

EXERCISE 5 (c)

(Section A)

1. Find the volume of a cylinder of which height $=7$ metres and radius $=10$ metres.
2. Find the curved surface of the cylinder of which height $=14$ metres and radius $=10$ metres.
3. The diameter of a cylindrical tank is 24·5 metres and depth 32 metres. How many metric tonnes of water will it hold ? (One cubic metre of water weighs 1000 kg).
4. Find the volume of a cylinder which has a height of 21 cm and a base of radius 5 cm. Also find the curved surface of the cylinder.
5. (*a*) How many cubic metres of earth must be dug out to make a well 20 metres deep and 2 metres in diameter ? $\left(\text{Take } \pi \text{ to be } \frac{22}{7}\right)$

   (*b*) If the inner curved surface of the well in part (*a*) above is to be plastered at the rate of Rs. 5 per sq. metre, find the cost of plastering. $\left(\text{Take } \pi \text{ to be } \frac{22}{7}\right)$
6. The area of the curved surface of a cylinder is 4400 cm^2, and the circumference of its base is 110 cm.

   Find (*i*) the height of the cylinder,

   (*ii*) the volume of the cylinder. $\left(\text{Take } \pi \text{ to be } \frac{22}{7}\right)$
7. A cylinder has a diameter of 20 cm. The area of the curved surface is 1000 cm^2. Find (*i*) the height of the cylinder correct to one decimal place, (*ii*) the volume of the cylinder correct to one decimal place. (Take π to be 3·14)

(Section B)

8. Find the volume of a hollow cylinder (open at both ends) whose external diameter is 44 dm, thickness 2 dm, and height 25 dm.
9. A cylindrical vessel, whose base is 14 dm in diameter holds 2310 litres of water. Taking a litre of water to occupy 1000 cubic cm, what is the height of the vessel in dm ?

10. Find in cubic dm the material in a cylindrical tube, the radius of the outer surface being 10 dm, the thickness 4 dm and the height 9 dm.

11. The volume of a right circular cylinder is 1100 cubic cm and the radius of its base is 5 cm, find the area of its curved surface.

12. Find the whole surface of a hollow cylinder open at the ends, if its length is 8 cm the external diameter is 10 cm and the thickness is 2 cm. (Take $\pi=3{\cdot}1416$)

13. A cylindrical tube open at both ends is made of metal. The internal diameter of the tube is 11·2 cm, and its length is 21 cm. The metal everywhere is 0·4 cm. Calculate the volume of the metal correct to one place of decimal. $\left(\text{Take } \pi \text{ to be } \frac{22}{7}\right)$.

14. Find how many pieces of money $\frac{3}{4}$ cm in diameter and $\frac{1}{8}$ cm thick must be melted down to form a cube whose edge is 3 cm long.

15. The radius of the inner surface of a lead pipe is 1·5 dm and the radius of the outer surface is 1·9 dm. If the pipe be melted and formed into a solid cylinder of the same length as before, find its radius.

(Section C)

16. A cubic metre of iron is to be down into a cylindrical wire 50 cm in diameter. What is the length of the wire to the nearest centimetre ?

17. A well with 10 metres inside diameter is dug 14 metres deep. Earth taken out of it has been spread all-round it to a width of 5 metres to form an embankment. Find the height of the embankment.

18. How many litres of water flow through a pipe in one minute if the bore (diameter) of the pipe is 12 cm and water flows at the rate of 3·5 kilometres an hour, and 1 cubic decimetre of water measures 1 litre ?

19. Water flows along a pipe of radius 0·6 cm at 8 cm per second. This pipe is draining the water from a tank which holds 1000 litres of water when full. How long would it take to completely empty the tank ?

20. Water flows through a cylindrical pipe of internal diameter 7 cm at 5 metres per second (ms^{-1}).

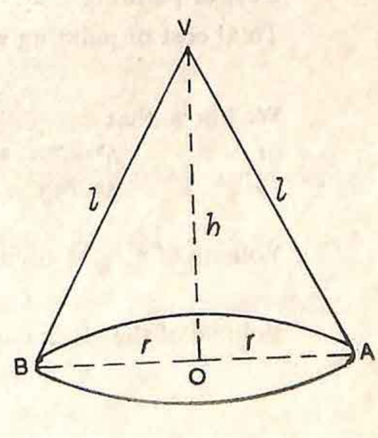
Calculate (*a*) the volume, in litres, of water discharged by the pipe in one minute ;

(*b*) the time, in minutes, the pipe would take to fill an empty rectangular tank 4 m×3 m×23·1 m. $\left(\text{Take } \pi \text{ to be } \frac{22}{7}\right)$

5·5. CONE

You have seen a joker's cap or a ice cream cone. Each of these objects has the shape of a **cone.**

A cone is a solid pointed figure with a circular base. It has one vertex, one edge, one plane surface and one curved surface. In the figure, V is the vertex, O is the centre of the circular base and OA is its radius. We will deal with a right circular cone in which line segment VO is perpendicular to the base.



The length of the segment VO is called the **height** of the cone and is denoted by h.

The distance of the vertex from any point on the circumference of the base is called the **slant height** of the cone. It is denoted by l.

By Pythagoras theorem, we have $r^2+h^2=l^2$

A cone is generally defined as the solid generated by the revolution of a right-angled triangle about one of its sides, containing right angle.

The side of the triangle about which the triangle is revolved is called the **axis** of the cone. The hypotenuse side is called the **generator** of the cone. The third side forms the circular base and is the **radius** of the base.

We state without proof the following formulae :

Volume of a right circular cone $=\frac{1}{3}\pi r^2 h$

Curved surface of a right circular cone $=\pi rl$

Whole surface of a right circular cone $=$(curved surface)$+$(area of the base)

$=\pi rl+\pi r^2$ $\quad=\pi r(l+r)$

We know that the volume of a cylinder of radius r and height h is $\pi r^2 h$. If a cone of height h and radius r is constructed, its volume is $\frac{1}{3}\pi r^2 h$.

What do you observe ?

The volume of a cone is *one-third* of the volume of a cylinder having the same base and the same radius.

A cross-section of a cone by a plane through its axis is an *isosceles triangle*. But, a cross-section of a cone by a plane parallel to the base is a *circle*.

Example 7. The diameter of a right circular cone is 14 dm and its slant height is 12 dm. Find the cost of painting its whole surface at the rate of 15 P per square dm. Also find its volume.

Solution. Here radius of the base $(r)=\frac{14}{2}$ dm$=7$ dm

Slant height $(l)=12$ dm

Curved surface of a right circular cone$=\pi rl$

$\therefore$ Curved surface of the given cone $=\frac{22}{7}\times 7\times 12$ sq. dm

$=264$ sq. dm

Area of the base of the cone $=\pi r^2$

$=\frac{22}{7}\times 7\times 7$ sq. dm

$=154$ sq. dm

Whole surface area of the cone $=(264+154)$ sq. dm

$=418$ sq. dm

Cost of painting 1 sq. dm $=15$ P

Total cost of painting whole surface of the cone

$=15\times 418$ P$=6270$ P $\quad$ **=Rs. 62·70**

We know that $\quad r^2+h^2=l^2$

or $\quad h^2=l^2-r^2 \quad =144-49$

or $\quad h^2=95$

$\therefore \quad h=\sqrt{95}=9{\cdot}75$ dm

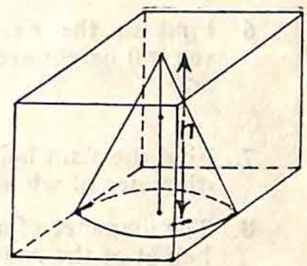
Volume of a right circular cone $=\frac{1}{3}\pi r^2 h$

Volume of the given cone $=\frac{1}{3}\times\frac{22}{7}\times 7\times 7\times 9{\cdot}75$ cubic dm

$=500{\cdot}5$ cubic dm.

Example 8. Find the volume of the largest right circular cone that can be cut out of a cube whose edge is 9 cm.

Solution. The base of the cone will be the circle inscribed in a face of the cube and its height will be equal to an edge of the cube.



Edge of the cube$=9$ cm
For the cone,
Radius of the base $(r)=\frac{9}{2}$ cm
Height (h) $=9$ cm
Volume of a right circular cone

$$=\frac{1}{3}\pi r^2 h$$

Volume of the cone to be cut out

$$=\frac{1}{3}\times\frac{22}{7}\times\frac{9}{2}\times\frac{9}{2}\times 9 \text{ cubic cm}$$

$$=\frac{2673}{14} \text{ cm}^3 \qquad =\mathbf{190{\cdot}93\ cm^3} \text{ nearly.}$$

Example 9. A right circular cylinder and a right circular cone have equal bases and equal heights. If their curved surfaces are in the ratio 8 : 5, show that the radius of their base is to their height as 3 : 4.

Solution. Let r and h be the base-radius and height of the cylinder and cone.

Curved surface of the cylinder$=2\pi rh$
Curved surface of the cone $=\pi rl$ where $l=\sqrt{r^2+h^2}$
By the question, $2\pi rh : \pi rl = 8 : 5$
or $5\times 2\pi rh = 8\times\pi rl$
or $5h=4l$
or $5h=4\sqrt{r^2+h^2}$
Squaring both sides, $25h^2=16(r^2+h^2)$
or $9h^2=16r^2$
Taking square root, $3h=4r$
or $\frac{r}{h}=\frac{3}{4}$

i.e., radius of the base : height$=3:4$.

EXERCISE 5 (d)

(Section A)

1. Find the curved surface of a right cone whose slant height is 25 cm and radius is 7 cm.
2. Calculate the curved surface area of a cone whose perpendicular height is 4·8 cm and the radius of whose base is 3·6 cm. Leave your answer in terms of π.
3. Find the volume of a right circular cone whose base-radius is 8 cm and vertical height is 14 cm.
4. Find the volume of a right circular conical tent whose vertical height is 8 m and the area of whose base is 156 m^2.

5. The height of a circular cone is 36 cm and diameter of its base is 21 cm. Find the whole surface of the cone.

6. Find to the nearest cubic centimetre the volume of a cone whose slant height and vertical height are 5 cm and 4 cm respectively.

(Section B)

7. Find the slant height of a cone whose volume is equal to 12,936 cubic metres and the diameter of whose base is 42 metres.

8. The diameter of a cone is 21 cm. Its volume is 1848 cubic cm. Find the perpendicular height of the cone.

9. The volume of a cone is 616 cubic metres. Its perpendicular height is 27 metres. Find the radius of the base.

10. The area of the curved surface of a right circular cone of diameter 14 cm is 550 sq. cm. Find its volume.

11. How many square metres of canvas are required for a conical circus tent whose height is 35 metres and the radius of the base 84 metres? Also find the volume of air contained in it.

12. A conical tent is required to accommodate 7 persons. Each person requires 22 sq. dm of space on the floor and 176 cubic dm of air to breathe. Find the vertical height, the slant height and width of the tent.

13. The volume of a cone is the same as that of a cylinder whose height is 9 cm. and diameter 40 cm. Find the radius of the base of the cone, if its height is 108 cm.

$\left(\text{Take } \pi \text{ to be } \dfrac{22}{7}\right)$

14. A cylindrical vessel of internal diameter 2 cm has twice the capacity of a conical vessel of internal radius 1·5 cm and internal depth 4 cm. Find the height of the cylinder.

15. The radius and the height of a right circular cone are in the ratio 5 : 12. If its volume is 314 cubic metres, find the slant height and the radius.

(Use $\pi = 3{\cdot}14$)

(Section C)

16. A conical vessel whose internal radius is 10 centimetres and height 48 centimetres is full of water. If this water is poured into a cylindrical vessel with internal radius 20 centimetres, find the height to which the water rises in it.

17. From a solid right circular cylinder with height 10 cm and radius of the base 6 cm, a right circular cone of the same height and base is removed. Find the volume of the remaining solid.

18. A girl fills a cylindrical bucket 32 cm in height and 18 cm in radius with sand. She empties the bucket on the ground and makes a conical heap of the sand. If the height of the conical heap is 24 cm, find (*i*) the radius, and (*ii*) the slant height of the heap. Give your answers correct to one place of decimals.

19. From a solid cylinder whose height is 8 cm and radius 6 cm, a conical cavity of height 8 cm and of base radius 6 cm is hollowed out. Find the volume of the remaining solid correct to one place of decimal.

$\left(\text{Take } \pi \text{ to be } \dfrac{22}{7}\right)$

5·6. SPHERE

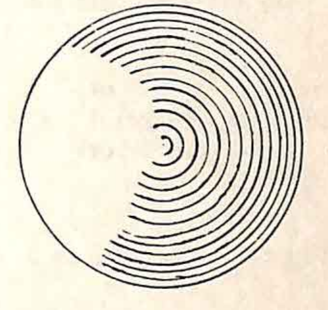
Glass marbles used for play, small metal balls used in cycle bearings, iron shot used for shot-put are some examples of a **sphere**. A tennis ball is a hollow sphere.

A sphere is a set of points in space whose distance from a fixed point is equal to a given distance.

The fixed point is called the **centre** and the given distance is called the **radius** of the sphere.

A line segment passing through the centre of the sphere having its end points on the sphere is called a **diameter** of the sphere.

All diametres of a sphere are of constant length, each being equal to twice the radius of the sphere.



A plane containing the diameter of a sphere divides the sphere into two equal parts. Each part is called a **hemi-sphere.** A lemon cut into two halves gives two hemi-spheres.

A sphere is also defined as the solid generated by the rotation of a semi-circle about its diameter. The centre and radius of the semi-circle are respectively the centre and radius of the sphere.

The following formulae are stated without proof :

Volume of a sphere of radius $r=\frac{4}{3}\pi r^3$

Surface area of a sphere of radius $r=4\pi r^2$

Volume of a hemisphere of radius $r=\frac{2}{3}\pi r^3$.

Example 10. Find the surface and volume of a sphere whose radius is 21 dm.

Solution. Radius of the sphere$=21$ dm

Surface area of a sphere$=4\pi r^2$

$\therefore$ Surface area of the sphere$=4\times\frac{22}{7}\times 21\times 21$ sq. dm

$=$**5,544 sq. dm**

Volume of a sphere $=\frac{4}{3}\pi r^3$

$\therefore$ Volume of the sphere $=\frac{4}{3}\times\frac{22}{7}\times 21\times 21\times 21$ sq. dm^3

$=$**38,808 cubic dm**

Example 11. A hemisphere of lead of radius 6 cm is cast into a right circular cone of height 75 cm. Find the radius of the base of the cone.

Solution. Radius of the hemisphere $(r)=6$ cm

Volume of the hemisphere $=\frac{2}{3}\pi r^3$

$=\frac{2}{3}\times\frac{22}{7}\times 6^3$ cubic cm

Let the radius of the base of the cone be x cm.

Height of the cone $=75$ cm

Volume of the cone $=\frac{1}{3}\pi r^2 h$

$=\frac{1}{3}\times\frac{22}{7}\times x^2\times 75$ cubic cm

Now the volume of the cone$=$the volume of the hemisphere

$$\therefore \quad \frac{1}{3}\times\frac{22}{7}\times x^2\times 75=\frac{2}{3}\times\frac{22}{7}\times 6^3$$

or $$x^2\times 75=2\times 6\times 6\times 6$$

or $$x^2=\frac{2\times 6\times 6\times 6}{75}$$

$$=\frac{144}{25}$$

$$\therefore \quad x=\frac{12}{5}$$

$$=\mathbf{2{\cdot}4\ cm}.$$

Example 12. The diameter of a sphere is 6 cm. It is melted and drawn into a wire of diameter 0·2 cm. Find the length of the wire. [*C.B.S.E., 1983* (*Delhi*)]

Solution. Diameter of the sphere = 6 cm

$\therefore$ Radius of the sphere = 3 cm

Volume of a sphere $=\frac{4}{3}\pi r^3$

$\therefore$ Volume of the given sphere $=\frac{4}{3}\times\pi\times 3\times 3\times 3\ \text{cm}^3$

$=36\pi\ \text{cm}^2$

Volume of wire to be drawn = Volume of given sphere

$=36\pi\ \text{cm}^3$

Radius of wire = 0·1 cm

Let the length of drawn wire be l cm.

Volume of a cylinder $=\pi r^2 l$

$\therefore$ Volume of drawn wire $=\pi\times 0{\cdot}1\times 0{\cdot}1\times l\ \text{cm}^3$

$=0{\cdot}01\pi l\ \text{cm}^2$

But the volume of drawn wire $=36\pi\ \text{cm}^3$

$\therefore \quad 0{\cdot}01\pi l=36\pi$

or $\quad \frac{1}{100}l=36$

i.e., $\quad l=3600$ cm

$=$ **36 metres.**

EXERCISE 5 (e)

(Section A)

1. Find the surface and volume of a sphere whose radius is 10·5 dm.
2. The diameter of a spherical shot-put, made of brass, is 14 cm. Find its surface area and the volume.
3. The total surface area of a sphere is 3850 sq. cm. Find the diameter of the sphere.
4. The volume of a sphere is 38,808 cubic cm. What is the radius of the sphere ?

5. How many litres of water will a hemispherical bowl contain whose radius is 0·7 metre ? (1 litre = 1000 cubic cm).

(Section B)

6. A solid metal sphere is cut through its centre into two equal parts. If the diameter of the sphere is 3·5 cm, find the total surface of each part, correct to two decimal places. $\left(\text{Take } \pi = \frac{22}{7}\right)$

7. The volume of one sphere is 64 times that of another sphere. Calculate the ratio of their (*i*) radii, (*ii*) surface area.

8. How many bullets can be made out of a cube of lead whose edge measures 22 cm, each bullet being 2 cm in diameter ?

9. How many (spherical) lead shots each 0·3 cm in diameter can be made from a rectangular solid 9 cm by 11 cm by 12 cm ?

10. The radius of the base of a cone is 4 cm ; find the height so that the volume may be equal to that of a sphere with diameter 4 cm.

11. A metallic disc, in the shape of a right circular cylinder, is of height 2·5 mm, and base radius $2\sqrt{3}$ cm. 12 metallic discs are melted and made into a sphere. Calculate the radius of the sphere.

12. The radius of the base of a cone and the radius of a sphere are the same, each being 8 cm. Given that the volumes of these two solids are also the same. Calculate the slant height of the cone, correct to one place of decimal.

13. A spherical cannon ball, 6 cm in diameter, is melted and cast into a conical mould, the base of which is 12 cm in diameter. Find the height of the cone.

14. A sphere of diameter 6 cm is dropped into a cylindrical vessel partly filled with water. The diameter of the vessel is 12 cm. If the sphere is completely submerged, by how much will the surface of the water be raised ?

(Section C)

15. Find the volume of a solid in the form of a right circular cylinder with hemispherical ends whose extreme length is 24 dm and diameter 2·5 dm.

16. Marbles of diameter 1·4 cm, are dropped into a beaker containing some water and are fully submerged. The diameter of the beaker is 7 cm. Find how many marbles have been dropped in it, if the water rises by 5·6 cm.

17. What is the least number of solid metallic spheres of 6 cm in diameter that should be melted and recast to form a solid metal cylinder whose height is 45 cm and diameter 4 cm ?

18. A spherical ball of lead, 6 cm in diameter is melted and recast into three spherical balls. The diameters of two of these three balls are 3 cm and 4 cm respectively. Find the diameter of the third one.

19. A solid sphere of radius 6 cm is melted into a hollow cylinder of uniform thickness. If the external radius of the base of the cylinder is 4 cm and if its height is 72 cm, find the uniform thickness of the cylinder.

20. A vessel is in the form of an inverted cone. Its height is 8 cm, and the radius of its top which is open is 5 cm. It is filled with water up to the rim. When lead shots, each of which is a sphere of radius 0·5 cm, are dropped into the vessel, one-fourth of the water flows out. Find the number of lead shots dropped into vessel. $\left(\text{Take } \pi \text{ to be } \frac{22}{7}\right)$

REVIEW EXERCISES—IV

(Section A)

1. State for each of the following statements whether it is True or False :

(*a*) The area of a triangle with sides measuring a, b, c is given by $\sqrt{s(s-a)(s-b)(s-c)}$, where s is the perimeter of the triangle.

(*b*) The area of a triangle with base 4 cm and perpendicular height 6 cm is 24 cm^2.

(*c*) The volume of a rectangular solid measuring 1 m by 50 cm by 0·5 m is 2,50,000 cm^3.

(*d*) The volume of a sphere of radius r is given by the formula $4\pi r^3$.

(*e*) The volume of a cone is one-half of the volume of the cylinder of the same radius and height.

2. Fill in the blanks to make the following statements true :

(*a*) The slant height (l) of a right circular cone with height 12 cm and radius of the base 5 cm is......... [*C.B.S.E., 1981 (A.I.)*]

(*b*) The surface (area) A of a sphere is......... [*C.B.S.E., 1978 (Delhi)*]

(*c*) The volume of a sphere is given by......... [*C.B.S.E., 1977 (Delhi)*]

(*d*) If the radius of the base of a cylinder is doubled, its volume becomes.........

(*e*) In a cuboid, any three co-terminal edges are mutually.........

(*f*) The total surface of a right circular cone is given by the formula S=.........

(*g*) The volume of a cylinder of radius R and height H is given by......... [*C.B.S.E., 1987 (A.I.)*]

(*h*) If a cylinder and a cone are of the same height and have bases of equal radii, then the volume of the cylinder is............times that of the cone. [*C.B.S.E., 1987 (Delhi)*]

(*i*) The volume of a cone of height h and radius of the base r is......... [*C.B.S.E., 1986 (A.I.)*]

(*j*) Length of the longest rod that can be placed in a room of dimensions 6 metres, 6 metres and 3 metres is......... [*C.B.S.E., 1986 (Delhi)*]

3. The perimeter of a rectangle is 640 cm and the length is to the breadth as 5 : 3. Find its area.

4. What is the area of a square whose diagonal is 15 metres ?

5. The surface of a cube is 384 sq. cm. Find its volume.

6. How many times will the wheel of a car rotate in a journey of 38 km, if it is known that the diameter of the wheel is 56 cm ? $\left(\text{Take } \pi \text{ to be } \frac{22}{7}\right)$

7. A sphere of radius r has the same volume as that of a cone with a circular base of radius r. Find the height of the cone. [*C.B.S.E., 1984 (Delhi)*]

8. The radii of two cylinders are in the ratio 2 : 3 and their heights are in the ratio 5 : 3. Calculate the ratio of their volumes. [*C.B.S.E., 1981 (A.I.)*]

9. A piece of metal in the form of a cone of radius 3 cm and height 7 cm is melted and cast into a cube. Find a side of the cube. $\left(\text{Assume } \pi=\frac{22}{7}\right)$ [*C.B.S.E., 1981 (Delhi)*]

(Section B)

10. A floor which measures 15 m$\times$8 m is to be laid with tiles measuring 50 cm$\times$25 cm. Find the number of tiles required. Further, if a carpet is laid on the floor so that a space of 1 m exists between its edges and the edges of the floor, what fraction of the floor is uncovered ?

11. One side of a right-angled triangle is 126 metres, the difference between the hypotenuse and the other side is 42 metres. Find the remaining sides.

12. The side of a rhombus is ?6 cm and one of its diagonals is 18 cm, find the other diagonal and the area of the rhombus.

13. Find the area of a trapezium whose parallel sides are 11 metres and 25 metres long and the non-parallel sides are 15 metres and 13 metres long respectively.

14. The area of a circle is 154 sq. cm. Find the length of the side of the inscribed square.

15. A square and a rectangle, each have a perimeter of 48 metres. If the difference between the areas of the two figures is 4 sq. metres, what are the dimensions of the rectangle? [*C.B.S.E., 1983 (Delhi)*]

16. Three metal cubes with edges 6 cm, 8 cm and 10 cm respectively are melted together and formed into a single cube. Find the diagonal of this cube. [*C.B.S.E., 1981 (A.I.)*]

17. Find what length of canvas 2 metres in width is required to make a conical tent 12 metres in diameter and 6·3 metres in slant height. Also find the cost of the convas at the rate of Rs. 12·50 per metre. $\left(\text{Assume } \pi=\frac{22}{7}\right)$ [*C.B.S.E., 1983 (A.I.)*]

18. A hollow cylindrical tube open at both ends is made of iron 2 cm thick. If the external diameter be 50 cm and the length of the tube be 140 cm, find the number of cubic cm of iron in it.

19. A right-angled triangle, of which the sides are 3 dm and 4 dm in length is made to turn round on the longer side; find the volume of the cone thus formed.

20. A cone is 8·4 cm high and the radius of its base is 2·1 cm. It is melted and recast into a sphere. Determine the radius of the sphere.

21. A right circular cone is 4·1 cm high and the radius of its base is 2·1 cm. Another right circular cone is 4·3 cm high and the radius of the base is 2·1 cm. Both the cones are melted together and recast into a sphere. Determine the diameter of the sphere. [*C.B.S.E., 1982 (Delhi)*]

22. How many metres of wire 0·4 mm in diameter may be drawn from the amount of copper required to mould a solid sphere of diameter 18 cm?

(Section C)

23. If h, c, V are respectively the height, the curved surface and the volume of a cone, prove that

$$3\pi Vh^3-c^2h^2+9V^2=0$$

24. A solid cuboid with areas of adjacent faces 72 sq. cm, 36 sq. cm and 18 sq. cm respectively is melted and cast into a cube. Find the surface area of the cube. [*C.B.S.E., 1986 (Delhi)*]

25. A rectangular reservoir with a base 46 metres by 33 metres contains water 2 metres deep. In how much time will the reservoir be emptied by a pipe whose cross-section is a circle with radius 10 dm, if water is flowing at the rate of 7 kilometres per hour?

26. A solid metal sphere 6 cm in diameter, is formed into a cylindrical tube 10 cm in external diameter and 4 cm in length; find the thickness of the tube.

27. A spherical shell of lead whose external diameter is 18 cm is melted into a right circular cylinder, 8 cm high and 12 cm in diameter. Find the inner diameter of the shell.

28. The circular ends of a bucket are of radii 35 cm and 14 cm and the height of the bucket is 40 cm. Find the volume of the bucket. [*C.B.S.E., 1986 (Delhi)*]

□□

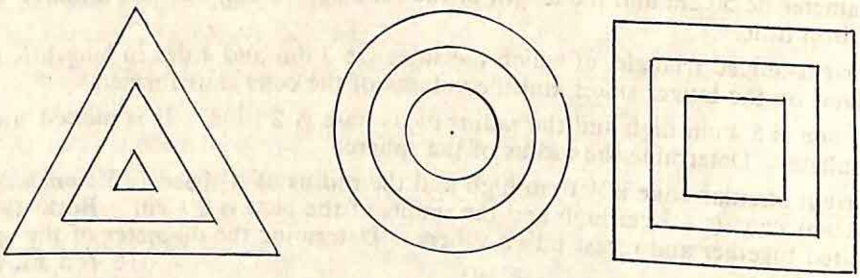
6

SIMILAR TRIANGLES

6·1. SIMILARITY

You have already learnt about congruent figures *i.e.*, figures which have same shape and same size. Some figures, however, have same shape but not necessarily the same size. Such figures are called **similar figures**. A map of a flat region, for example, is similar to the territory which it represents. A photograph can be enlarged and the enlargement is similar to the original.

Study the following figures :



You can see that any two equilateral triangles are similar, any two squares are similar. Mathematically, however, we must be more precise in defining the similarity of the closed plane figures.

Two rectilinear figures are said to be equiangular to one another, if the angles of the first, taken in order, are respectively equal to the angles of the second, taken in order.

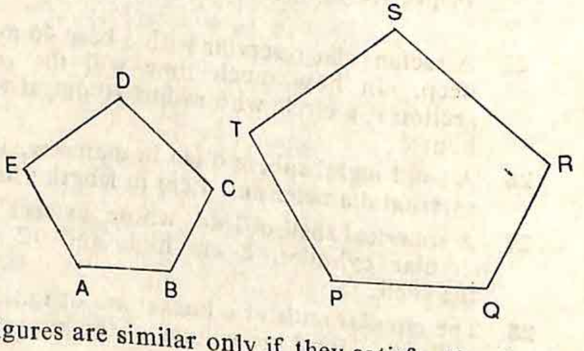
Two rectilinear figures are said to be similar, if they are equiangular to one another, and also their corresponding sides are proportional.

In these polygons $\angle A$ and $\angle P$, $\angle B$ and $\angle Q$, $\angle C$ and $\angle R$, $\angle D$ and $\angle S$, $\angle E$ and $\angle T$ are pairs of *equal* angles. Sides AB and PQ, BC and QR, CD and RS, DE and ST, EA and TP are *corresponding* sides.

Polygons ABCDE and PQRST are similar, if

(*i*) $\angle A = \angle P$, $\angle B = \angle Q$, $\angle C = \angle R$, $\angle D = \angle S$ and $\angle E = \angle T$

and (*ii*) $\dfrac{AB}{PQ} = \dfrac{BC}{QR} = \dfrac{CD}{RS} = \dfrac{DE}{ST} = \dfrac{EA}{TP}$

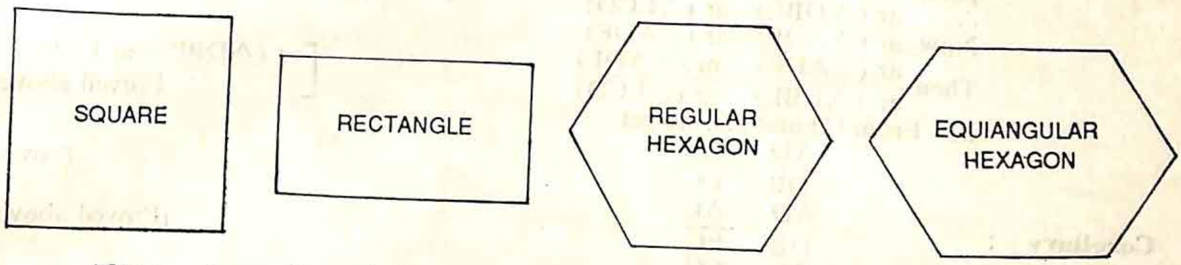


It should be noted carefully that two figures are similar only if they satisfy the following two conditions :

(*i*) *their angles are equal, each to each, and*

(*ii*) *their corresponding sides are proportional.*

If two polygons have only their angles equal, each to each, they need *not* to be similar. See figures given below :



If two polygons have their corresponding sides proportional, they need *not* be similar. For example, a rhombus and a square have their corresponding sides proportional, but they are not similar because their corresponding angles are not equal.

In case of triangles, the two conditions for similarity are *not* independent. Two triangles are similar if one condition is satisfied.

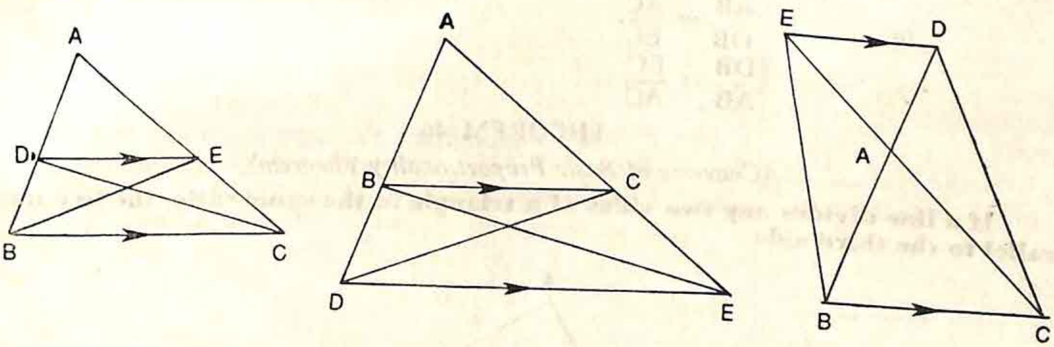
6·2. SIMILAR TRIANGLES

Before we discuss various characteristic properties of similar triangles, we prove the following basic results on proportionality.

THEOREM 45

(*Basic Proportionality Theorem*)

A line drawn parallel to one side of a triangle divides the other two sides in the same ratio. [*C.B.S.E., 1982 (Delhi) ; 1983 (A.I.) ; 1986 (A.I.)*]



Given : $\triangle ABC$.

A line DE is drawn parallel to BC meeting AB and AC (or these sides produced) at D and E respectively.

To Prove : $\dfrac{AD}{DB} = \dfrac{AE}{EC}$.

Const. : Join BE and CD.

Proof : $\triangle ADE$ and $\triangle DBE$ have a common vertex E and collinear bases.

$$\therefore \frac{\text{ar}(\triangle ADE)}{\text{ar}(\triangle DBE)} = \frac{AD}{DB} \qquad ...(1)$$

$\triangle ADE$ and $\triangle ECD$ have a common vertex D and collinear bases.

$$\therefore \frac{\text{ar}(\triangle ADE)}{\text{ar}(\triangle ECD)} = \frac{AE}{EC} \qquad ...(2)$$

$\triangle$DBE and $\triangle$ECD stand on the same base DE and lie between the same parallels.

$\therefore$ ar ($\triangle$DBE)=ar ($\triangle$ECD)

Now ar ($\triangle$ADE)=ar ($\triangle$ADE)

Then $\dfrac{\text{ar}(\triangle ADE)}{\text{ar}(\triangle DBE)}=\dfrac{\text{ar}(\triangle ADE)}{\text{ar}(\triangle ECD)}$ [ar ($\triangle$DBE)=ar ($\triangle$ECD) Proved above.

$\therefore$ From (1) and (2), we get

$$\frac{AD}{DB}=\frac{AE}{EC}$$

Proved.

Corollary : $\dfrac{AD}{DB}=\dfrac{AE}{EC}$ [Proved above.

or $\dfrac{DB}{AD}=\dfrac{EC}{AE}$

$\therefore$ $1+\dfrac{DB}{AD}=1+\dfrac{EC}{AE}$

or $\dfrac{AD+DB}{AD}=\dfrac{AE+EC}{AE}$

or $\dfrac{AB}{AD}=\dfrac{AC}{AE}$

Then $\dfrac{AD}{AB}=\dfrac{AE}{AC}$

Again $\dfrac{AD}{DB}=\dfrac{AE}{EC}$

$\therefore$ $\dfrac{AD}{DB}+1=\dfrac{AE}{EC}+1$

or $\dfrac{AD+DB}{DB}=\dfrac{AE+EC}{EC}$

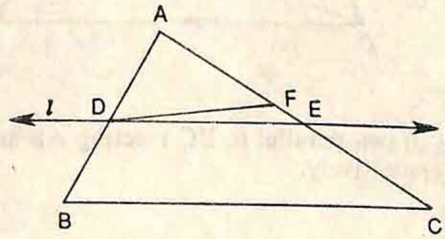
or $\dfrac{AB}{DB}=\dfrac{AC}{EC}$

$\therefore$ $\dfrac{DB}{AB}=\dfrac{EC}{AC}$

THEOREM 46

(*Converse of Basic Proportionality Theorem*)

If a line divides any two sides of a triangle in the same ratio, the line must be parallel to the third side.



Given : $\triangle$ABC in which line l intersects AB in D and AC in E such that

$$\frac{AD}{DB}=\frac{AE}{EC}$$

To Prove : $l \parallel$ BC

Proof : Suppose line l is not parallel to BC.

Let us draw through D a line DF paral lel to BC.

Now in $\triangle$ABC, DF $\parallel$ BC.

By Basic Proportionality Theorem, we get

$$\frac{AD}{DB}=\frac{AF}{FC}$$

But $\frac{AD}{DB}=\frac{AE}{EC}$ [Given.

$\therefore$ $\frac{AF}{FC}=\frac{AE}{EC}$

Then $\frac{AF}{FC}+1=\frac{AE}{EC}+1$

or $\frac{AF+FC}{FC}=\frac{AE+EC}{EC}$

or $\frac{AC}{FC}=\frac{AC}{EC}$

$\therefore$ $FC=EC$

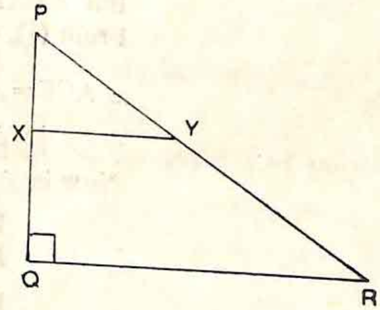
But this is impossible unless the points E and F coincide *i.e.*, DF is the line l itself.

Hence $l \parallel BC$. Proved.

EXERCISE 6 (a)

(Section A)

1. In a $\triangle ABC$, DE is drawn parallel to BC cutting the other sides at D and E.
   (*a*) If AD=12 cm, DB=8 cm and AE=9 cm, find EC.
   (*b*) If AB=10 cm, AC=7·5 cm, and AD=6 cm, calculate the length of AE and EC.
2. In a $\triangle ABC$, DE is drawn parallel to BC cutting the other sides produced at M and N.
   (*a*) If AB=27 mm, AC=21 mm and AM=36 mm, find the length of AN.
   (*b*) If AM=6·6 cm, MB=2·4 cm and AC=4·9 cm, calculate the length of NC.
3. Prove that the line drawn through the mid-point of one side of a triangle parallel to another side bisects the third side.



4. In the given figure PQR is a right-angled triangle, right-angled at Q, XY is parallel to QR, PQ=6 cm, PY=4 cm and PX : XQ=1 : 2. Calculate the lenghts of PR and QR.

(Section B)

5. Prove that the line joining the middle points of two sides of a triangle is parallel to the third side.
6. If there are three or more parallel lines, and the intercepts on any line that cuts them are equal, then prove that the corresponding intercepts on any other line that cuts them are also equal.
7. PQ is drawn parallel to BC cutting the other two sides of a $\triangle ABC$ at P and Q respectively such that AB=24 mm, BC=35 mm and CA=28 mm.
   (*a*) If P divides AB internally in the ratio 5 : 3, calculate the length of the segments AP, PB, AQ and QC.
   (*b*) If P divides AB externally in the ratio 5 : 3 find the length of the segments AP, PB, AQ and QC.

8. Prove that any line drawn from the vertex of a triangle to the base is bisected by the line which joins the middle points of the other two sides of the triangle.

9. X, Y, Z are the mid-points of the sides QR, RP, PQ of $\triangle PQR$. Show that PZXY is a parallelogram.

10. If the middle points of the adjacent sides of any quadrilateral are joined, show that the figure thus formed is a parallelogram.

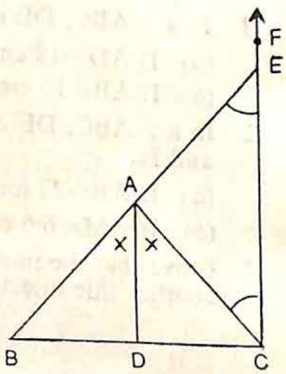
(Section C)

11. Prove that a line drawn parallel to the parallel sides of a trapezium cuts the non-parallel sides proportionally.

12. Prove that the diagonals of a trapezium cut each other proportionally.

13. Prove that the bisector of an angle of a triangle divides the opposite side in the ratio of the sides containing the angle.

Solution :



Given : $\triangle ABC$ in which AD, the bisector of $\angle A$ meets BC at D.

To prove : $\frac{BD}{DC}=\frac{AB}{AC}$

Const. : From C, draw ray CF $\parallel$ DA meeting BA produced in E.

Proof : DA $\parallel$ CE and AC cuts them,

$\angle DAC$=alternate $\angle ACE$ ...(1)

Again, DA $\parallel$ CE and BE cuts them,

$\angle BAD$=corresponding $\angle AEC$ ...(2)

But $\angle BAD=\angle DAC$ [Given ...(3)

From (1), (2) and (3), we have

$\angle ACE=\angle AEC$

$\therefore$ In $\triangle CAE$, AE=AC [Opposite sides,

Now in $\triangle CBE$, DA $\parallel$ CE

$$\therefore \quad \frac{BD}{DC}=\frac{BA}{AE}$$

or $$\frac{BD}{DC}=\frac{AB}{AC}$$ [AE=AC proved.

14. If a line drawn from the vertex of a triangle divides the opposite internally in the ratio of the sides containing the angle, prove that the line bisects the angle.

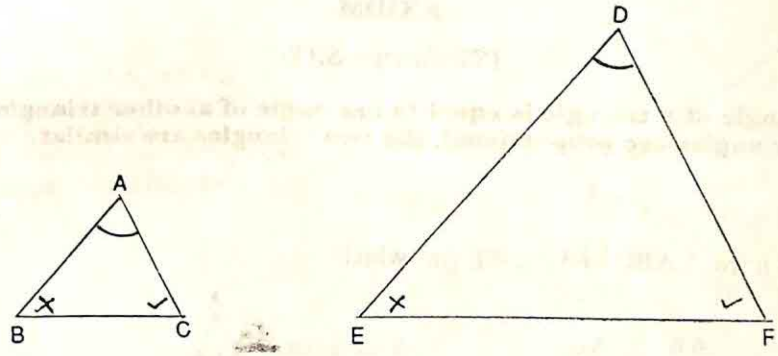
AXIOM

(Similarity—AAA)

If in two triangles, the corresponding angles are equal i.e. if the two triangles are equiangular, then the two triangles are similar.

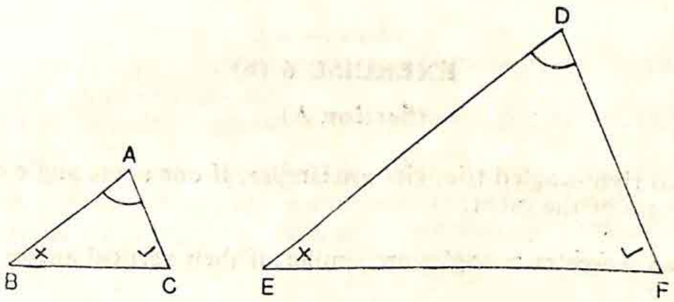
Here we have $\triangle ABC$ and $\triangle DEF$ in which

$\angle A=\angle D$, $\angle B=\angle E$ and $\angle C=\angle F$



Hence $\triangle ABC \sim \triangle DEF$

Corollary : If two angles of one triangle are equal to two angles of another, each to each, the two triangles are similar.



Here we have $\triangle ABC$ and $\triangle DEF$ in which $\angle B = \angle E$ and $\angle C = \angle F$.

Then $\angle A + \angle B + \angle C = \angle D + \angle E + \angle F = 180°$

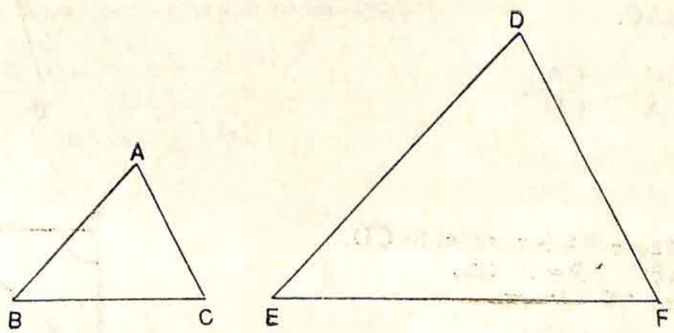
$\therefore \quad \angle A = \angle D$

So, by AAA similarity, the two triangles—$\triangle ABC$ and $\triangle DEF$ are similar.

AXIOM

(Similarity—SSS)

If the three sides of one triangle are proportional to the three sides of another, the triangles are similar.



Here we have $\triangle ABC$ and $\triangle DEF$ such that

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

Hence $\triangle ABC$ and $\triangle DEF$ are similar.

AXIOM

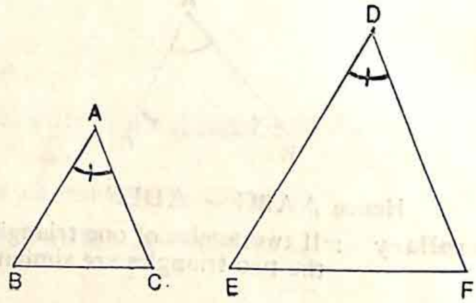
(Similarity—SAS)

If one angle of a triangle is equal to one angle of another triangle and the sides including these angles are proportional, the two triangles are similar.

Here we have $\triangle ABC$ and $\triangle DEF$ in which $\angle A = \angle D$

and $\dfrac{AB}{DE} = \dfrac{AC}{DF}$.

Hence $\triangle ABC$ and $\triangle DEF$ are similar.



EXERCISE 6 (b)

(Section A)

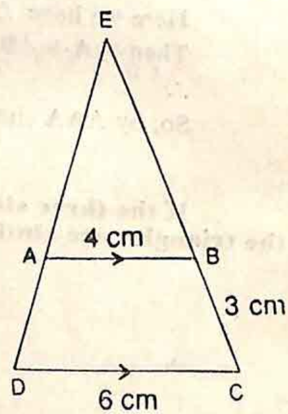
1. Prove that two right-angled triangles are similar, if one acute angle of the one is equal to an acute angle of the other.

2. Prove that two isosceles triangles are similar, if their vertical angles are equal.

3. In the figure alongside ABCD is a trapezium with AB parallel to DC. Given that AB=4 cm, BC=3 cm and CD=6 cm.

   (*a*) Name two triangles in the figure which are similar.

   (*b*) Calculate the length of EB.

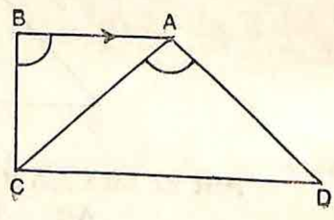


4. D is a point on the side BC of a $\triangle ABC$ such that $\angle ADC = \angle BAC$.

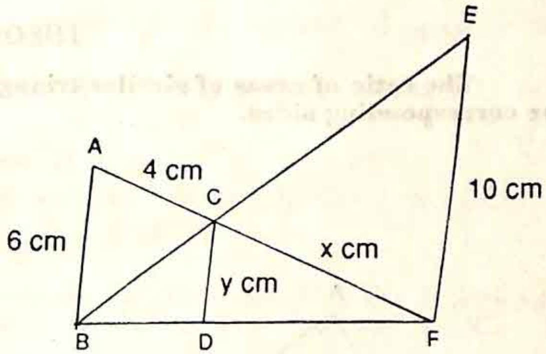
   Prove that $\dfrac{BC}{CA} = \dfrac{CA}{CD}$.

5. In the given figure BA is parallel to CD, $\angle DAC = \angle ABC$, AB=10 cm, BC=9 cm, and AC=15 cm.

   Calculate the length of AD.



6. In the adjoining figure, AB, CD and EF are parallel lines. Given that AB=6 cm, CD=y cm, EF=10 cm, AC=4 cm and CF=x cm, calculate the value of x and y.



(Section B)

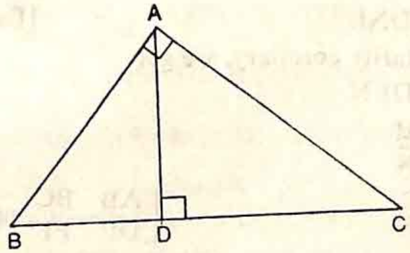
7. The base AB of an isosceles $\triangle ABC$ is produced both ways to D and E such that AD . BE=AC^2 ; show that $\triangle ACD$ and $\triangle BCE$ are similar.
8. In two equiangular triangles, show that the medians are proportional to the corresponding sides.
9. Prove that a line drawn parallel to one side of a triangle cuts off a triangle similar to the given triangle.

(Section C)

10. Prove that any line drawn parallel to the base of a triangle and meeting the two sides, is bisected by the median to the base.
11. If the diagonals of a quadrilateral intersect each other proportionately, prove that it is a trapezium.

THEOREM 47

A perpendicular drawn from the vertex of the right angle of a right-angled triangle divides the triangle into two triangles similar to each other and also to the original triangle.



Given : $\triangle ABC$ in which $\angle A=90°$
AD is drawn perpendicular to the hypotenuse BC.

To Prove : (*i*) $\triangle DBA \sim \triangle ABC$
(*ii*) $\triangle DAC \sim \triangle ABC$
(*iii*) $\triangle DBA \sim \triangle DAC$

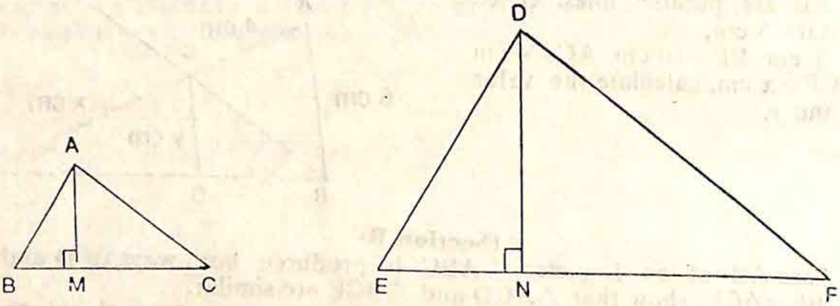
Proof : (*i*) In $\triangle DBA$ and $\triangle ABC$
$\angle ADB = \angle CAB$ [Each is a right-angle.
$\angle DBA = \angle ABC$ [Common angle.
So, by AA—Similarity corollary, we have
$\triangle DBA \sim \triangle ABC$

(*ii*) In $\triangle DAC$ and $\triangle ABC$
$\angle ADC = \angle BAC$ [Each being a right angle.
$\angle DCA = \angle ACB$ [Common angle.
So, by AA—Similarity corollary, we have
$\triangle DAC \sim \triangle ABC$

(*iii*) $\triangle DBA \sim \triangle ABC$ (Proved above)
$\triangle DAC \sim \triangle ABC$ (Proved above)
Hence $\triangle DBA \sim \triangle DAC$ Proved.

THEOREM 48

The ratio of areas of similar triangles is equal to the ratio of the squares on the corresponding sides. [*C.B.S E., 1980 (A.I.) ; 1986 (Delhi)*]



Given : Two similar triangles ABC and DEF.

To Prove : $\frac{ar\,(\triangle ABC)}{ar\,(\triangle DEF)}=\frac{AB^2}{DE^2}=\frac{BC^2}{EF^2}=\frac{AC^2}{DF^2}$.

Const. : Draw AM $\perp$ BC and DN $\perp$ EF.

Proof : $\triangle ABC$ and $\triangle DEF$ are similar. [Given.

$\therefore\ \angle A=\angle D,\ \angle B=\angle E$ and $\angle C=\angle F$

and $\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}$ ...(1)

In $\triangle ABM$ and $\triangle DEN$

$\angle B=\angle E$ [Proved above.

$\angle AMB=\angle DNE$ [Each being a right angle.

So, by AA—Similarity corollary, we get

$\triangle ABM \sim \triangle DEN$

$\therefore\ \frac{AB}{DE}=\frac{AM}{DN}$

$\therefore\ \frac{AM}{DN}=\frac{BC}{EF}$ $\left[\frac{AB}{DE}=\frac{BC}{EF}\right.$, proved above.

$\frac{ar\,(\triangle ABC)}{ar\,(\triangle DEF)}=\frac{\frac{1}{2}BC.AM}{\frac{1}{2}EF.DN}$ [Area of a triangle $=\frac{1}{2}$ base $\times$ alt.

$=\frac{BC}{EF}\cdot\frac{AM}{DN}$

$\frac{ar\,(\triangle ABC)}{ar\,(\triangle DEF)}=\frac{BC}{EF}\cdot\frac{BC}{EF}=\frac{BC^2}{EF^2}$ $\left[\frac{AM}{DN}=\frac{BC}{EF}\right.$ proved above.

But $\frac{BC}{EF}=\frac{AC}{DF}=\frac{AB}{DE}$ [Proved above.

$\therefore\ \frac{ar\,(\triangle ABC)}{ar\,(\triangle DEF)}=\frac{AB^2}{DE^2}=\frac{BC^2}{EF^2}=\frac{AC^2}{DF^2}$ [Proved.

EXERCISE 6 (c)

(Section A)

1. The areas of two similar triangles are equal. Prove that they are congruent.
2. Two isosceles triangles have equal vertical angles and their areas have the ratio 9 : 16. Compare their heights.

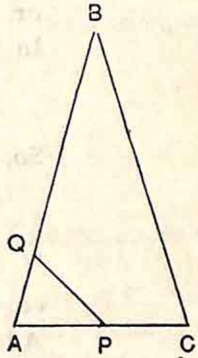
3. The areas of two similar triangles ABC and PQR are 64 sq. cm and 121 sq. cm respectively. If QR=15·4 cm, find BC.

4. ABC is a triangle, right-angled at A, and AD is drawn perpendicular to BC. Show that $\triangle BAD : \triangle ACD = BA^2 : AC^2$.

(Section B)

5. OAB and OCD are two lines through O such that OA=2 cm, AB=4 cm, OC=3 cm and CD=6 cm.
Prove that $\triangle OAC = \frac{1}{8}$ of the quad. ABCD.

6. ABC is a triangle and PQ is a line meeting AB in P and AC in Q. If AP=1 cm, PB=3 cm, AQ=1·5 cm and QC=4·5 cm, prove that $\triangle APQ = \frac{1}{16}$ of the $\triangle ABC$.



7. The given diagram shows two isosceles triangles which are similar. If PQ and BC are not parallel and PC=4, AQ=3, QB=12 and BC=15.
Calculate (*a*) the length of AP.
(*b*) the ratio of the areas of $\triangle APQ$ and $\triangle ABC$, assuming that AP=PQ.

8. (*a*) AD and BE are the medians of $\triangle ABC$ which meet in G. If DE is joined, compare the areas of the triangles ABG and DGE.
(*b*) Also prove that $\triangle DEG = \frac{1}{4}\triangle ABG = \frac{1}{12}\triangle ABC$.

9. Show that the areas of similar triangles are proportional to the squares on the corresponding (*i*) altitudes, (*ii*) medians.

10. ABC is a right-angled triangle and AD is perpendicular to the hypotenuse BC. Prove that
(*a*) $AB^2 = BD.BC$ (*b*) $AC^2 = BC.CD$ (*c*) $AD^2 = BD.DC$.

(Section C)

11. In $\triangle ABC$, $AD \perp BC$ and is such that $AD^2 = BD.DC$; prove that $\angle A$ is a right angle.

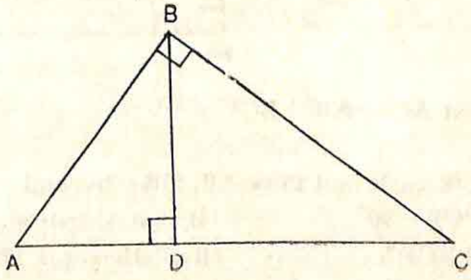
12. D, E, F are the middle points of the sides BC, CA and AB respectively of a $\triangle ABC$. Prove that $\triangle DEF = \frac{1}{4}\triangle ABC$.

THEOREM 49

(Pythagoras Theorem)

In a right-angled triangle, the square described on the hypotenuse is equal to the sum of the squares described on the sides containing the right-angle.

[*C.B.S.E.*, *1983* (*A.I.*) ; *1986* (*Delhi*)]



Given : A right-angled triangle ABC in which $\angle B=90°$.

To Prove : $AC^2=AB^2+BC^2$

Const. : Draw $BD \perp AC$.

Proof : In $\triangle ADB$ and $\triangle ABC$

$\angle ADB=\angle ABC$ [Each angle is 90°.

$\angle BAD=\angle CAB$ [Common angle.

So, by AA—Similarity corollary,

$\triangle ADB \sim \triangle ABC$

$\therefore \quad \frac{AD}{AB}=\frac{AB}{AC}$

or $\quad AB^2=AD.AC$ ...(1)

In $\triangle BDC$ and $\triangle ABC$

$\angle BDC=\angle ABC$ [Each angle is 90°.

$\angle DCB=\angle BCA$ [Common angle.

So, by AA—Similarity corollary,

$\triangle BDC \sim \triangle ABC$

$\therefore \quad \frac{DC}{BC}=\frac{BC}{AC}$

or $\quad BC^2=DC.AC$ ...(2)

Adding (1) and (2), we get

$AB^2+BC^2=AD.AC+DC.AC$

$=(AD+DC).AC$

$=AC.AC$

Hence $AB^2+BC^2=AC^2$

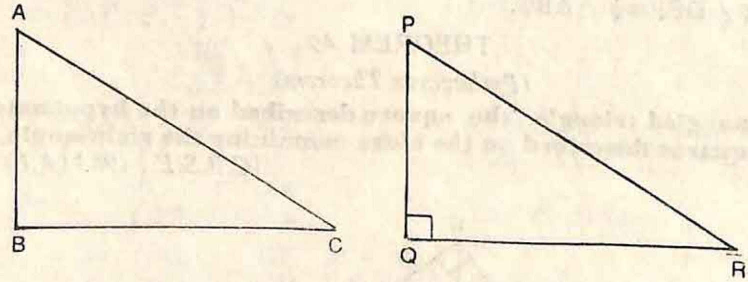
i.e., $\quad AC^2=AB^2+BC^2$ Proved.

THEOREM 50

(*Converse of Pythagoras Theorem*)

In a triangle, if the square of one side is equal to the sum of the two sides, then the angle opposite to the first side is a right angle.

[*C.B.S.E., 1981* (*Delhi*) ; *1985* (*A.I.*) ; *1987* (*Delhi*)]



Given : $\triangle ABC$ such that $AC^2=AB^2+BC^2$

To Prove : $\triangle ABC=90°$

Const. : Construct $\triangle PQR$ such that $PQ=AB$, $QR=BC$ and $\angle PQR=90°$

Proof : In $\triangle PQR$, $\angle PQR=90°$ [By Construction.

$\therefore \quad PR^2=PQ^2+QR^2$ [By Pythagoras Theorem.

or $PR^2=AB^2+BC^2$ [$PQ=AB$, $QR=BC$, by Construction.

But $AC^2=AB^2+BC^2$ [Given.

$\therefore$ $PR^2=AC^2$

or $PR=AC$

Now in $\triangle ABC$ and $\triangle PQR$

$AB=PQ$ [By Construction.

$BC=QR$ [By Construction.

$AC=PR$ [Proved above.

$\therefore$ $\triangle ABC \cong \triangle PQR$ [SSS—Congruency Theorem.

Then $\angle B=\angle Q$ [Corresponding parts of congruent triangles.

But $\angle Q=90°$ [By Construction.

Hence $\angle B=90°$ Proved.

EXERCISE 6 (d)

(Section A)

1. A ladder l metres long reaches the base of a window of a house h metres above the ground. How far is the foot of the ladder from the house ?

2. A ladder 13 m long rests against a vertical wall. If the foot of the ladder is 5 m from the foot of the wall, find the distance of the other end of the ladder from ground.

3. Prove that the square on the diagonal of a given square is twice the given square.

4. In a right-angled triangle ABC, it is given that the hypotenuse $AC=2{\cdot}5$ cm, and the side $AB=1{\cdot}5$ cm. Calculate the side BC.

5. Prove that the three times the square on any side of an equilateral triangle is equal to four times the square on its altitude.

6. In $\triangle ABC$, $\angle BCA$ is a right-angle and Q is mid-point of BC.
Prove that $BC^2=4(AQ^2-AC^2)$

(Section B)

7. The diagonals of a rhombus are 8 cm and 6 cm. Find the perimeter of the rhombus.

8. A ladder 50 dm long is placed so as to reach, a window 48 dm high ; and on turning the ladder over to the other side of the street, it reaches a point 14 dm high. Calculate the breadth of the street.

9. Prove that the sum of the squares on the diagonals of a rhombus is equal to the sum of the squares on the sides.

10. A point O is taken inside a $\triangle ABC$; OP, OQ, OR are drawn perpendicular to BC, CA and AB respectively. Show that

$$BP^2+CQ^2+AR^2=CP^2+AQ^2+BR^2$$

11. In $\triangle ABC$, $\angle C$ is acute and $AD \perp BC$.
Prove that $AB^2=BC^2+AC^2-2\,BC.CD$

12. In $\triangle ABC$, $\angle C$ is obtuse and $AD \perp BC$.
Prove that $AB^2=BC^2+AC^2+2\,BC.CD$

(Section C)

13. In $\triangle ABC$, $\angle C=1$ right angle, M is the mid-point of BC and N of CA.
Prove that $4\,(AM^2+BN^2)=5\,AB^2$

14. If ABCD be a rectangle and O any point within it, show that

$$OA^2+OC^2=OB^2+OD^2$$

REVIEW EXERCISE V

(Section A)

1. In answer to each of the following statements write True or False as appropriate :

(*a*) If two triangles are congruent, then they are also similar.

(*b*) The altitude on the hypotenuse of a right-angled triangle is the mean proportional between the segments of the hypotenuse.

(*c*) If the sides of a triangle measures 3 cm, 4 cm, 5 cm, then it is a right triangle.

2. Fill in the blanks to make the following statements true :

(*a*) The ratio of any two corresponding sides of two similar triangles is 3 : 2. The ratio of the areas of the two triangles is.................... [*C.B.S.E., 1984 (A.I.)*]

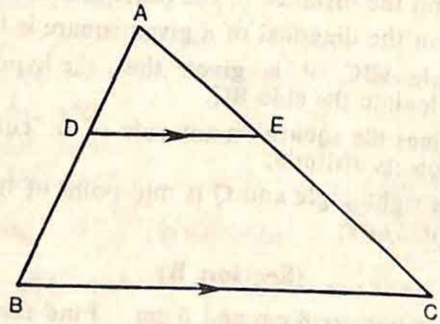
(*b*) If in two triangles corresponding angles are equal, their corresponding sides are........... [*C.B.S.E., 1983 (A.I.)*]

(*c*) The ratio of areas of similar triangles is equal to the ratio of the.................on the corresponding sides. [*C.B.S.E., 1982 (Delhi); 1986 (A.I.)*]

(*d*) If a line divides two sides of a triangle in the same ratio, then the line is............. to the third side. [*C.B.S.E., 1980 (A.I.); 1986 (A.I.)*]

(*e*) If PQ is drawn parallel to side BC of a triangle ABC, where AP : PB=1 : 3 and AQ=4 cm, then AC=........... [*C.B S.E., 1986 (Delhi)*]

(*f*) If the corresponding sides of two triangles are proportional, then the triangles are................. [*C.B.S.E., 1986 (Delhi)*]



(*g*) In the figure, DE ∥ BC, $\frac{AD}{DB}=\frac{3}{5}$.
If AC=4·8 cm, then AE=.............. [*C.B.S.E., 1984 (Delhi)*]

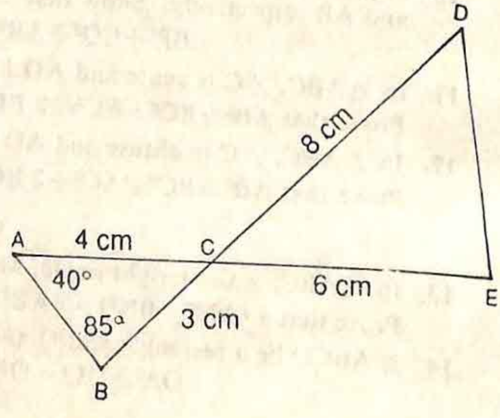
(*h*) If two polygons have their corresponding sides proportional, they..............be similar.

3. In the given figure ACE and BCD are two lines, ∠A=40° and ∠B=85°. Using the measurements given in the figure, complete the following true statements :

(*i*) Triangles ABC and CDE are similar because.........

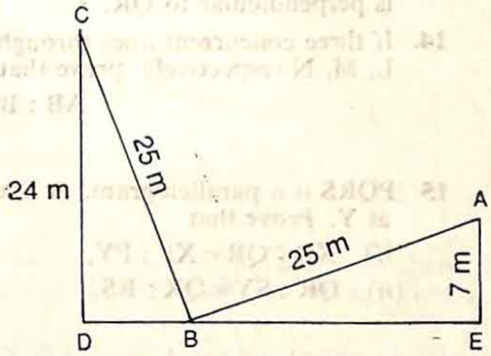
(*ii*) The size of ∠D is........

(*iii*) If AB=*x* cm, then ED=........



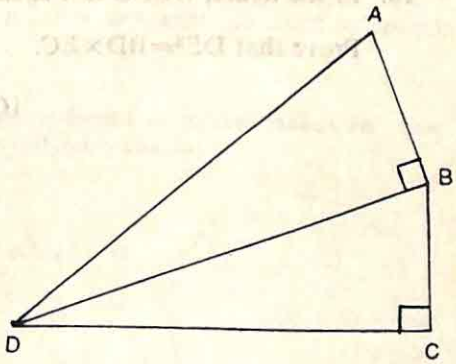
(Section B)

4. In the given figure, AB=BC=25 m. If AE=7 m, and CD=24 m, find the length of DE and also show that $\triangle ABE$ and $\triangle BDC$ are congruent.



5. The given figure ABCD represents a quadrilateral in which AD=13 cm, DC=12 cm, BC=3 cm and $\angle ABD=\angle BCD=90°$.

Calculate the length of AB.



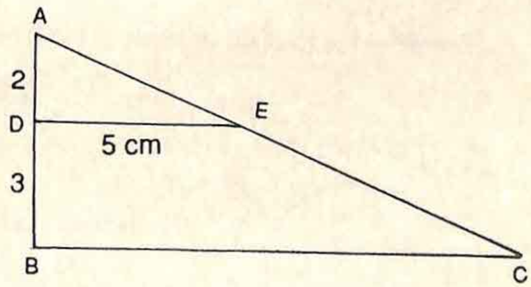
6. ABC is a triangle in which $\angle BAC=90°$ and $AD \perp BC$. Prove that $AD^2=BD \times DC$. [*C.B.S.E., 1982* (*Delhi*); *1985* (*A.I.*)]

7. In $\triangle ABC$, $AD \perp BC$ and $AD^2=BD \times DC$. Prove that ABC is a right triangle. [*C.B.S.E., 1983* (*Delhi*)]

8. In $\triangle ABC$, right-angled at C, Q is the mid-point of the side BC, show that $AB^2=4AQ^2-3AC^2$. [*C.B.S.E., 1981* (*Delhi*)]

9. P and Q are points on the sides CA and CB respectively of a triangle ABC, right angled at C.
Prove that $AQ^2+BP^2=AB^2+PQ^2$
[*C.B.S.E., 1982* (*A.I.*) ; *1984* (*Delhi*) ; *1984* (*A.I.*)]

10. In the figure given alongside $\triangle ABC \sim \triangle ADE$. If AD : DB=2 : 3, and DE=5 cm, (*i*) find BC. (*ii*) if x be the length of the perpendicular from A to DE, find the length of the perpendicular from A to BC in terms of x.



11. In $\triangle ABC$, points D, E, F are the mid-points of BC, CA and AB respectively. Prove that AD bisects EF.

12. ABC is a triangle, D is a point on AB such that $AD=\frac{1}{4}AB$ and E is a point on AC such that $AE=\frac{1}{4}$ AC. Prove that $DE=\frac{1}{4}$ BC.

13. ABCD is a rhombus with P, Q, R as a mid-points of AB, BC and CD. Prove that PQ is perpendicular to QR.

14. If three concurrent lines through P are cut by two parallel transversals in A, B, C and L, M, N respectively, prove that

$$AB : BC = LM : MN$$

(Section C)

15. PQRS is a parallelogram. A line through R meets PQ produced at X and PS produced at Y. Prove that

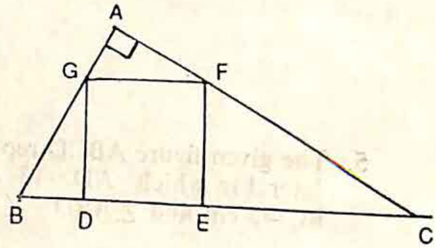
(*i*) XQ : QR = XP : PY,

(*ii*) QR : SY = QX : RS.

16. In the figure, DEFG is a square and $\angle BAC = 90°$.

Prove that $DE^2 = BD \times EC$.

[*C.B.S.E*, *1987* (*A.I.*)]



□ □

7

CIRCLES

7·1. CIRCLE

We come across many objects in our daily life which are round *i.e.*, circular in shape. The word '*circular*' means '*like a circle*' or '*of the shape of a circle*'. We shall now discuss circles and some of their properties.

A circle is a set of those points in a plane which are at a given constant distance from a given fixed point in the plane.

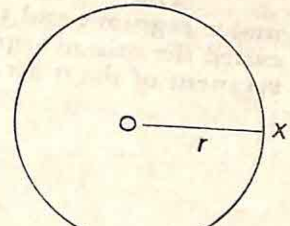
The fixed point is called the **centre** of the circle and the constant distance is called the **radius** of the circle.

Circle can also be defined as a locus.

If a point moves in a plane such that its distance from a given point in the plane remains constant, then the locus of the point is called a circle.

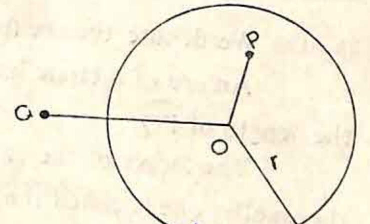
A circle with centre O and radius r is denoted by **C (O, r)**. In set notation we write as

$$C=\{X : OX=r\}$$



A point P is said to lie **inside** the circle when $OP<r$.

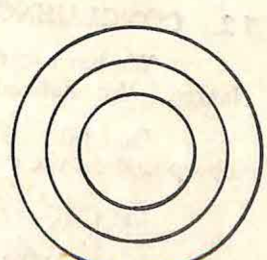
A point Q is said to lie **outside** the circle when $OQ>r$.



A circle of radius r divides the plane into three mutually disjoint sets—*interior*, *exterior and the circle itself*.

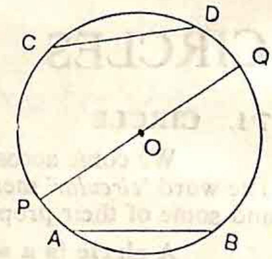
The perimeter of a circle is called its **circumference.**

Circles having the same centre are said to be **concentric circles**. Note that these circles have *different* radii.



Note that we have used the notation AB to denote the line segment AB and as also its length. Similarly the word 'radius' will be used for a line segment joining the centre to any point on the circle and also for its length.

A line segment joining two points on a circle is called a chord.



AB and CD are the two chords of the circle with centre O.

A line segment PQ passing through the centre O of the circle and having its end points P and Q on the circle is called a **diameter** of the circle.

Observe that *a chord passing through the centre of the circle is a diameter of the circle.*

Here PQ is a diameter and OP, OQ are two radii.

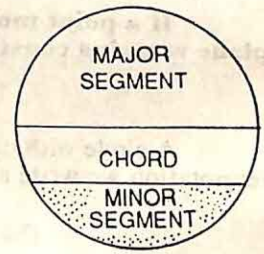
Then $\quad PQ = OP + OQ \quad = 2 \times \text{radius}.$

How many diameters can be had for a given circle ?

A circle has many diameters. All diameters of a circle are equal.

A chord of a circle divides the region enclosed by the circle into two parts. Each of the parts is called a **segment** of the circle.

The segment containing the diameter is called the major segment and the segment not containing the diameter is called the minor segment. Each of them is called the **alternate segment** of the other.

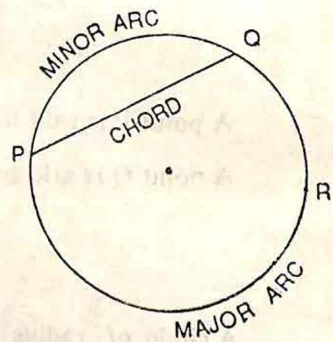


A chord of a circle divides the circle into two parts, each of which is called an **arc** of the circle.

We denote the arc from P to Q by $\overset{\frown}{PQ}$.

An arc of a circle has a length. Note that $l\,(\overset{\frown}{PQ})$ denotes the length of $\overset{\frown}{PQ}$.

The larger of the two arcs is called the **major arc** and the smaller one is called the **minor arc**. Here $\overset{\frown}{PQ}$ is the minor and arc $\overset{\frown}{PRQ}$ is the major arc.



The diameter of a circle divides the circle into *two equal parts*, each of which is called a **semi-circle.**

7·2. CONGRUENCE OF CIRCLES AND ARCS

We have learnt about congruence of line segments, angles and triangles in earlier classes. We shall now discuss congruence of circles and arcs.

Generally two circles are said to be congruent if and only if one of them can be superposed on the other so as to cover it exactly.

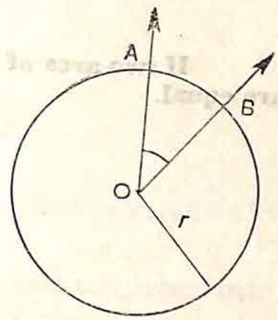
Let C (O, r) and C (O′, s) be two circles.

Let us superpose circle C (O′, s) on the circle C (O, r) so that O′ coincides with O.

It can be easily seen that C (O′, s) will cover C (O, r) completely, if and only if $s = r$.

Hence we can say that *two circles are congruent, if and only if, they have equal radii.*

Let C (O, r) be any circle.



Then any angle whose vertex is centre O is called a **central angle.**

Here $\angle AOB$ is the central angle. It intercepts the minor arc $\overset{\frown}{AB}$ of the circle.

The length of an arc is closely associated with the central angle determining the arc. So, the degree measure of an arc is defined in terms the central angle.

The degree measure of a circle is taken to be 360°. Then the degree measure of a semi-circle is 180°. The degree measure of an arc $\overset{\frown}{AB}$ is denoted by $m(\overset{\frown}{AB})$.

The degree measure of a minor arc is the measure of the central angle containing the arc.

The degree measure of a major arc is 360° minus the degree measure of the corresponding minor arc.

Two arcs of a circle (or of congruent circles) are congruent, if one of them can be superposed on the other so as to cover it completely. This is possible only when degree measures of the two arcs are equal.

Hence we can say that *two arcs of a circle (or of congruent circles) are said to be congruent, if and only if they have the same degree measure.*

Thus, if $m(\overset{\frown}{PQ})=m(\overset{\frown}{RS})$, then $\overset{\frown}{PQ}\cong\overset{\frown}{RS}$.

If $\overset{\frown}{AB}$ is congruent to $\overset{\frown}{CD}$, we write $\overset{\frown}{AB}\cong\overset{\frown}{CD}$.

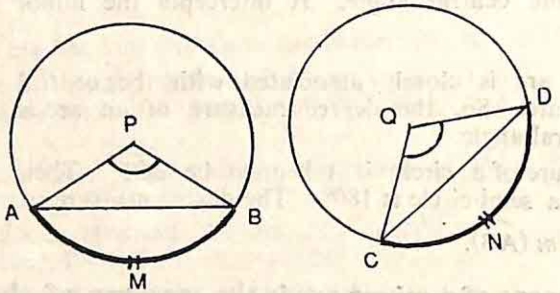
With every arc—minor or major is associated a unique chord obtained by joining the two ends of the arc.

EXERCISE 7 (a)

1. What is the distance of a diameter from the centre of circle ?
2. A cycle-wheel is lying flat on the ground. A long rod is placed over it. In how many places do they make contact ?
3. Do concentric circles have any common point ?
4. PQ is a chord of a circle whose centre is O. Is PO=QO ?
5. Is a major arc of a given circle greater than its semi-circle ?
6. What is the relation between a diameter and a radius of the circle ?
7. When is a chord, a diameter of the circle ?
8. What can be the maximum length of a chord of a circle ?
9. What is the difference between a circle and its circumference ?
10. If X is any point on the chord PQ of a circle with centre A, show that $AX \leqslant AP$.

THEOREM 51

If two arcs of congruent circles are congruent, then their corresponding chords are equal.



Given : Two congruent circles with centres P and Q.

$\widehat{AB} \simeq \widehat{CD}$

To Prove : $AB = CD$

Const. : Join AP, BP, CQ and DQ.

Proof : $\widehat{AB} \cong \widehat{CD}$ (Given)

$\therefore \quad m\,\widehat{AB} = m\,\widehat{CD}$

i.e., $\angle APB = \angle CQD$

In $\triangle APB$ and $\triangle CQD$

$PA = QC$ [Radii of congruent circles]

$PB = QD$ [Radii of Congruent circles]

Incl. $\angle APB$=Incl. $\angle CQD$ [Proved above]

$\therefore \quad \triangle APB \cong \triangle CQD$ [SAS Congruency Axiom]

$\therefore \quad AB=CD$ [Corresponding parts of congruent triangles]

Proved.

This theorem is also true if the congruent arcs are taken in the *same* circle. Then it can be stated as under :

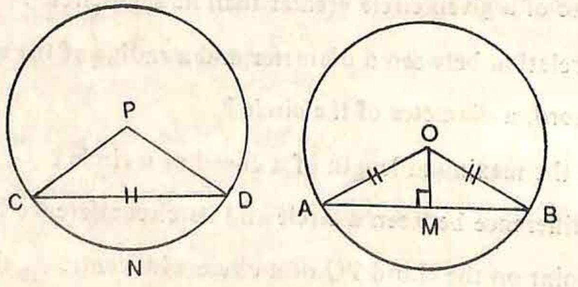
If two arcs of a circle are congruent, their corresponding chords are equal.

Its proof is similar to the above.

THEOREM 52

(*Converse of Theorem 51*)

If two chords of congruent circles are equal, then their corresponding arcs are congruent.



Given : Two congruent circles with centres O and P in which

$AB = CD$

To Prove : $\widehat{AB} \cong \widehat{CD}$

Const. : Let $\widehat{AB}$ and $\widehat{CD}$ be minor arcs. Join AO, BO, CP and DP.

Proof : In $\triangle AOM$ and $\triangle CPD$

$OA = PC$ [Radii of the congruent circles]

$AB = CD$ [Given]

$OB = PD$ [Radii of the congruent circles]

$\therefore \triangle AOM \cong \triangle CPD$ [SSS Congruency Theorem]

$\therefore \angle AOB = \angle CPD$ [Corresponding parts of congruent triangles

Then $m\widehat{AB} = m\widehat{CD}$

$\therefore \widehat{AB} \cong \widehat{CD}$ Proved

This theorem is also true when equal chords are taken in the *same* circle. Then] it can be stated as under :

If two chords of a circle are equal, their corresponding arcs are congruent.

Its proof is similar. Write the complete proof yourself.

EXERCISE 7 (b)

(Section A)

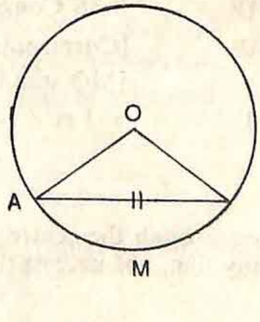
1. AB and AC are equal chords of a circle ; show that A is the mid-point of the arc BAC.
2. If two equal chords AB and CD of a circle intersect in it, prove that chord BC=chord AD.
3. If a pair of opposite sides of a quadrilateral inscribed in a circle are equal, prove that its diagonals are equal.

(Section B)

4. If two triangles are inscribed in congruent circles (or in the same circle) such that two sides of the one are equal to two sides of the other, each to each, prove that the triangles are congruent.
5. If in two circles two equal chords subtend equal angles at the centres, prove that the two circles are congruent.

THEOREM 53

The perpendicular drawn from the centre of a circle to a chord, bisects the chord. [*C.B.S.E., 1977 (Delhi)*]



Given : A circle with centre O.
chord AB, and OM ⊥ AB.

To Prove : AM=MB.

Const. : Join OA and OB.

Proof : In △OMA and △OMB

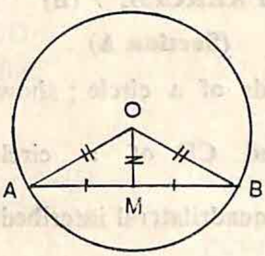
∠OMA=∠OMB =90°	[OM ⊥ AB, given]
hyp. OA=hyp. OB	[Radii of the same circle]
and OM=OM	[Common to both]
∴ △OMA≅△OMB	[RHS Congruency Theorem]
Then AM=MB	[Corresponding parts of congruent triangles]

Proved.

THEOREM 54

(*Converse of Theorem 53*)

A line drawn from the centre, of a circle to bisect a chord, which is not a diameter, is at right-angles to the chord.



Given : A circle with centre O.
Chord AB and its mid-point M.
OM is joined.

To Prove : OM ⊥ AB.

Const. : Join OA and OB.

Proof : In △sOMA and OMB

AM=MB	[Given]
OA=OB	[Radii of the same circle]
OM=OM	[Common to both]
△OMA≅△OMB	[SSS Congruency Theorem]
Then ∠OMA=∠OMB	[Corresponding parts of congruent triangles.]
∠OMA+∠OMB=2rt ∠s	[MO stands on AB]
∴ ∠OMA=∠OMB	=1 rt.∠.
i e., OM ⊥ AB	

Proved.

Note. When a chord passes through the centre it becomes a diameter. Its mid-point coincides with the centre. Hence any line, not necessarily perpendicular but passing through the centre, bisects it.

EXERCISE 7 (c)

(Section A)

1. Prove that the right-bisector of a chord of a circle passes through the centre of the circle.

2. In a circle of 5 cm radius, a chord 6 cm in length is placed. Find the distance of the chord from the centre.

3. A chord 4 cm in length is placed in a circle at a distance of 1·5 cm from the centre. Calculate the radius of the circle.

4. In a circle of a radius 5 cm, AB and CD are two parallel chords of length 8 cm and 6 cm respectively. Calculate the distance between the chords, if they are on

   (*i*) the same side of the centre.

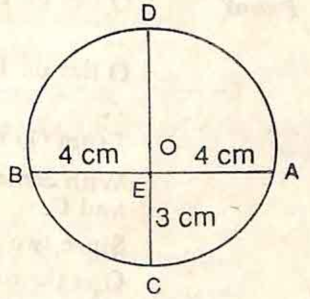
   (*ii*) opposite sides of the centre.

5. (*a*) Two parallel chords 3 cm and 4 cm in length respectively are placed on either side of the centre of a circle of radius 2·5 cm. Find the distance between them.

   (*b*) If they are placed on the same side of the centre, what will the distance be ?

(Section B)

6. In the figure given alongside, CD is a diameter which meets the chord AB in E, such that AE=BE=4 cm. If CE is 3 cm, find the radius of the circle.



7. Prove that the line joining the mid-points of two parallel chords of a circle (*a*) passes through the centre, (*b*) is perpendicular to both the chords.

8. ABCD is a rectangle whose vertices lie on the circumference of a circle. Prove that AC and BD are the diameters of the circle.

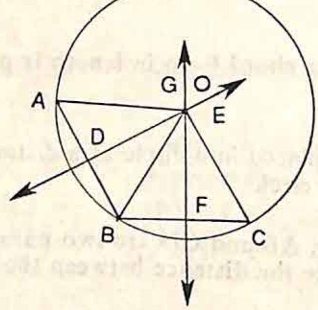
9. O is a centre of a circle of radius 5 cm. P is any point in the circle such that OP=3 cm. A is the point travelling along the circumference. x is the distance from A to P. What are the least and greatest values of x in cm ? What is the position of the points O, P and A at these values ?

(Section C)

10. If two circles intersect each other, then prove that the line joining their centres bisects the common chord at right-angles.

THEOREM 55

There is one circle, and only one, which passes through three given points, not in a line.



Given : Three non-collinear points A, B and C.

To Prove : There is one and only one circle passing through the points A, B and C.

Const. : Draw line segments AB and BC.

Draw DE and FG, the right-bisectors of AB and BC respectively.

Since A, B, C are not collinear, the right-bisectors of AB and BC are not parallel. They will intersect at some point O.

Join OA, OB and OC.

Proof : O lies on DE, the right-bisector of AB.

$\therefore$ OA=OB ...(1)

O lies on FG, the right-bisector of BC.

$\therefore$ OB=OC ...(2)

From (1) and (2), we get OA=OB=OC=r (say)

With centre O and r as radius one circle can be drawn to pass through A, B and C.

Since two lines DE and FG can intersect only at one point,

O is the only point equidistant from A, B and C.

Hence only one circle can be drawn through A, B and C.

i.e., there is a unique circle passing through three non-collinear points.

Proved.

EXERCISE 7 (d)

(Section A)

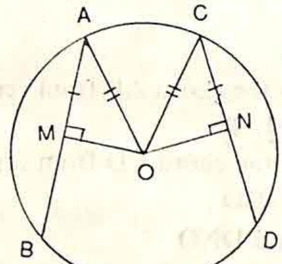
1. Given an arc of a circle. Complete it.
2. If from a certain point within a circle three or more equal line segments be drawn to the circumference, show that the point is the centre of the circle.
3. Prove that the right-bisectors of the sides of a triangle meet at a point.

(Section B)

4. Show that a circle can be drawn to pass through the angular points of a rectangle.
5. If the right-bisectors of the sides of a polygon are concurrent, show that a circle can be drawn through its vertices.

THEOREM 56

Equal chords of a circle are equidistant from the centre. [*C.B.S.E., 1978 (A.I).*]



Given : A circle with centre O in which chord AB=chord CD. $OM \perp AB$ and $ON \perp CD$.

To Prove : OM=ON.

Const. : Join OA and OC.

Proof : OM is prep. to the chord AB from centre O.

$\therefore$ $AM=\frac{1}{2}AB$

ON is prep. to the chord CD from centre O.

$\therefore$ $CN=\frac{1}{2}CD$

or $\frac{1}{2}AB=\frac{1}{2}CD$ [AB=CD, given]

Then AM=CN [$\frac{1}{2}AB=AM$ and $\frac{1}{2}CD=CN$ proved above]

In the rt. $\angle$d$\triangle$s AMO and CNO

hyp. OA=hyp. OC [Radii of the same circle]

AM=CN [Proved above]

$\therefore$ $\triangle AMO \cong \triangle CNO$ [RHS Congruency Theorem]

Then OM=ON [Corresponding parts of congruent triangles]

Proved

i.e., chord AB and CD are equidistant from the centre O.

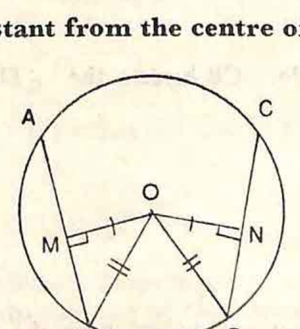
The above theorem is also true for *congruent* circles. It can be stated as :

In congruent circles, equal chords are equidistant from the centres.

THEOREM 57

(*Converse of Theorem 56*)

Chords which are equidistant from the centre of a circle are equal.



Given : Chords AB and CD of the circle with centre O.
$OM \perp AB$; $ON \perp CD$
and $OM = ON$.

To Prove : $AB = CD$

Const. : Join OB and OD.

Proof : OM is drawn perp. to the chord AB from centre O.

$\therefore$ $BM = \frac{1}{2}AB$

ON is drawn perp. to the chord CD from centre O.

$\therefore$ $DN = \frac{1}{2}CD$

In rt. $\angle d$ $\triangle$s BMO and DNO

hyp. OB = hyp. OD [Radii of the same circle]

$OM = ON$ [Given]

$\therefore$ $\triangle BMO \cong \triangle DNO$ [RHS Congruency Theorem]

Then $BM = DN$ [Corresponding parts of congruent triangles.]

i.e., $\frac{1}{2}AB = \frac{1}{2}CD$ [$BM = \frac{1}{2}$ AB and $DN = \frac{1}{2}$ CD, proved above]

$\therefore$ $AB = CD$ Proved.

This theorem is also true for congruent circles. It can be stated as :

In congruent circles, chords which are equidistant from the centre are equal.

EXERCISE 7 (e)

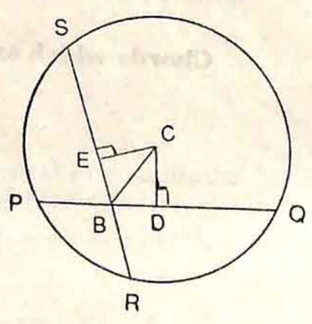
(Section A)

1. In a circle with centre O, two equal chords AB and AC are drawn. Prove that $\angle BAO = \angle CAO$.
2. AB and CD are two chords of a circle with M and N as their mid-points. Prove that MN makes equal angles with AB and CD.
3. Two equal chords of a circle intersect at a point in it. Prove that they are equally inclined to the diameter through the point of intersection.
4. AB and CD are equal chords of a circle whose centre is O. When produced, these chords meet at E. Prove that $EB = ED$ and $EA = EC$.

(Section B)

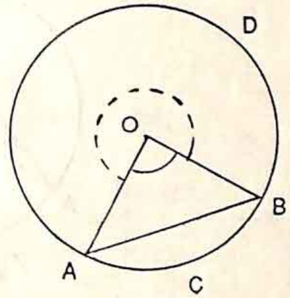
5. Draw a circle with diameter of 5 cm. Place in it two equal chords, each 3 cm long. Calculate their distances from the centre and verify your result by measurement.
6. C is the centre of the circle. CB bisects the $\angle DBE$, $CD \perp PQ$ and $CE \perp RS$.

   Prove that $PQ = RS$.



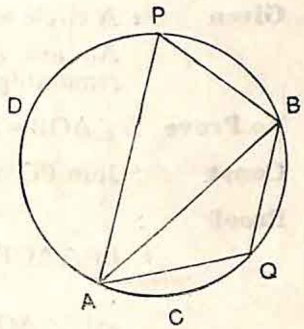
7. If two chords of a circle cut each other and make equal angles with a line which joins their point of intersection to the centre, prove that they are equal.

7·3 ANGLES IN CIRCLES



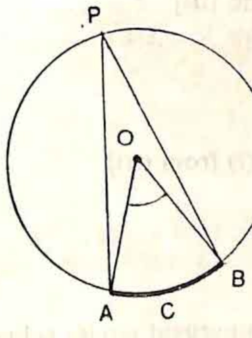
The angle *subtended* by the line segment AB *i.e.*, chord AB at O is $\angle$AOB, and the angle subtended by the arc ACB at O is also $\angle$AOB.

Also, arc ADB *subtends* the reflex angle AOB at the centre **O**.

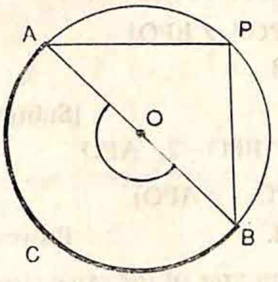


Chord AB *subtends* $\angle$APB at P and $\angle$AQB at Q. Also, arc ACB *subtends* $\angle$APB at P and arc ADB *subtends* $\angle$AQB at Q.

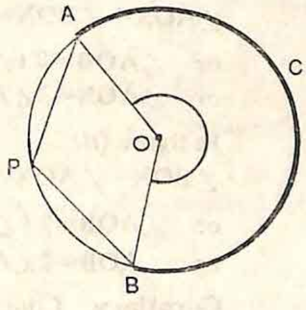
In the figures given below, arc ACB *subtends* $\angle$AOB at the centre O and $\angle$APB at the point P on the circumference.



(*i*)



(*ii*)



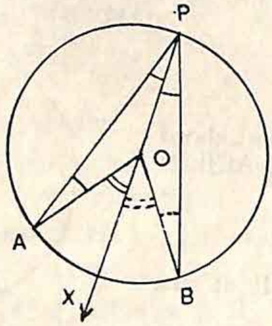
(*iii*)

$\angle$AOB is an acute angle in Fig. (*i*), a straight angle in Fig. (*ii*) and a reflex angle in (*iii*).

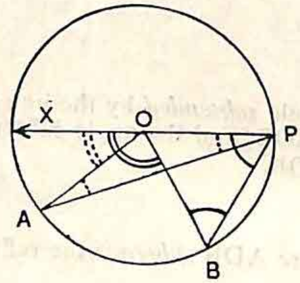
THEOREM 58

The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

[*C.B.S E., 1977 (A.I), 1980 (Delhi)*]



(i)



(ii)

Given : A circle with centre O.
An arc AB subtends $\angle AOB$ at the centre O and $\angle APB$ at a point P on the remaining part of the circumference.

To Prove : $\angle AOB = 2\angle APB$.

Const. : Join PO and produce it to any point X.

Proof :

In $\triangle AOP$, $OP = OA$ (Radii of the same circle.)

$\therefore$ $\angle OAP = \angle APO$ (Opposite angles to equal sides.)

ext. $\angle AOX = \angle OAP + \angle APO$ (ext. angle of a triangle=sum of int. opp. $\angle$s)

ext. $\angle AOX = 2\angle APO$ ...(i) ($\angle OAP = \angle APO$ Proved above)

Similarly in $\triangle BOP$,

ext. $\angle BOX = 2\angle BPO$ ...(ii)

In figure (i), [Adding (i) and (ii)]

$\angle AOX + \angle BOX = 2\angle APO + 2\angle BPO$

or $\angle AOB = 2(\angle APO + \angle BPO)$

or $\angle AOB = 2\angle APB$

In figure (ii), [Subtracting (i) from (ii)]

$\angle BOX - \angle AOX = 2\angle BPO - 2\angle APO$

or $\angle AOB = 2(\angle BPO - \angle APO)$

or $\angle AOB = 2\angle APB$. Proved

Corollary. Congruent arcs of the same circle or congruent circles subtend equal angles at the remaining part of the circumference.

EXERCISE 7 (f)

(Section A)

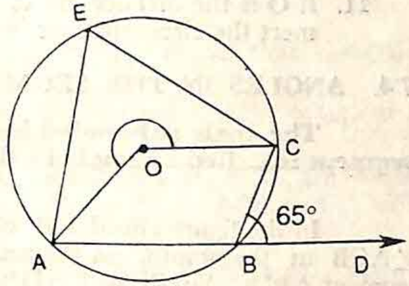
1. In a circle with centre O, there are two chords PQ and QR meeting at an angle of 68°. Find the magnitude of $\angle POR$.
2. Two radii OA and OB of a circle make an angle of 136° at the centre. If a point P lies on the minor arc so that $\angle OAP = 54°$, find the angle OBP.
3. O is the circumcentre of a $\triangle ABC$. D is the middle point of BC. Show that $\angle BOD = \angle A$.

4. Prove that parallel chords of a circle intercept equal arcs.

5. In the figure given alongside, O is the centre of the circle, ABD is a line and $\angle CBD=65°$.

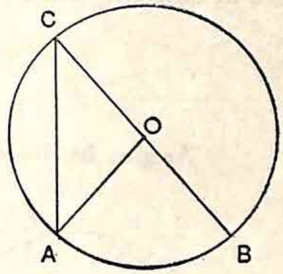
Find (*a*) $\angle AEC$

(*b*) $\angle AOC$ (angle marked)

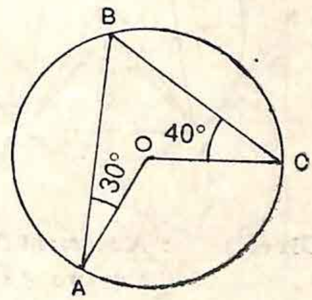


(Section B)

6. O is the centre of the circle. Given $\angle AOB=80°$. Calculate the value of $\angle OBA$ and $\angle OAC$.



7. In the given figure, O is the centre of the circle. $\angle OAB$ and $\angle OCB$ are 30° and 40° respectively. Find $\angle AOC$. Show your steps of working.

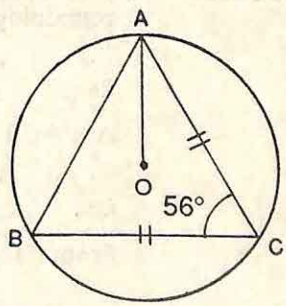


8. A, B, C, D are four points on the circumference of a circle in that order and AD=BC; show that AB is parallel to DC.

9. In the adjoining figure, O is the circumcentre of triangle ABC in which AC=BC. Given that $\angle ACB=56°$. Calculate

(*a*) $\angle CAB$

(*b*) $\angle OAC$.



10. Two equal circles intersect in A and B and through A a line is drawn to meet the circles in P and Q respectively. Prove that PB=BQ.

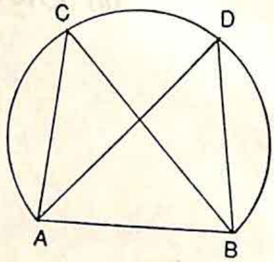
(Section C)

11. If O is the orthocentre of the triangle ABC and if the perpendicular AD is produced to meet the circumference in E, prove that OD=DE.

7·4. ANGLES IN THE SEGMENTS

The angle subtended by the base of a segment at any point on the arc of the segment is called an angle in the segment.

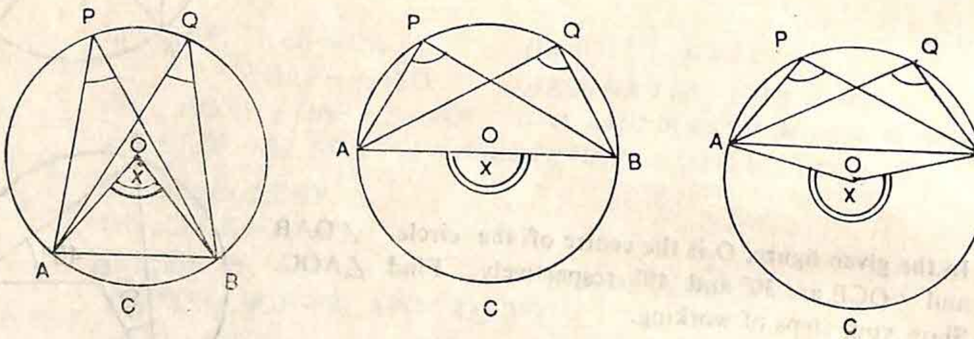
In the figure chord AB of the segment ABDC subtends $\angle ACB$ at the point C on the arc. Then $\angle ACB$ is an angle in the segment ABDC. Similarly $\angle ADB$ is an angle in the segment ABDC. Thus angles ACB and ADB are in the *same* segment.



Note : $\angle ACB$ and $\angle ADB$ are subtended by the same chord AB at different points C and D on the same arc ACDB.

THEOREM 59

Angles in the same segment of a circle are equal. [*C.B.S.E., 1981 (A I.)*]



Given : A segment APQB of a circle with centre O.
Any two $\angle$s APB and AQB in it.

To Prove : $\angle APB = \angle AQB$

Const. : Join AO and BO.

Proof : Arc ACB subtends $\angle AOB$ at the centre O and $\angle APB$ at a point P on the remaining part of the circumference.

$\therefore$ $\angle APB = \frac{1}{2} \angle AOB$

i.e., $\angle APB = \frac{1}{2} \angle x$ ...(*i*)

Arc ACB subtends $\angle AOB$ at the centre O and $\angle AQB$ on the circumference.

$\therefore$ $\angle AQB = \frac{1}{2} \angle AOB$

i.e., $\angle AQB = \frac{1}{2} \angle x$ ...(*ii*)

From (*i*) and (*ii*), we have

$\angle APB = \angle AQB$ Proved.

Corollary. Angles subtended by the same arc of a circle at different points on the circumference are equal.

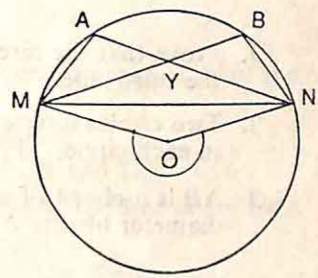
EXERCISE 7 (g)

(Section A)

1. AB and CD are two chords of a circle intersecting at O. Prove that $\triangle AOD$ and $\triangle BOC$ are equiangular.
2. In a circle with centre O, two chords AB and CD intersect each other at P. If $\angle ABC = 47°$ and $\angle APC = 113°$, find the angle BAD.
3. Through P, a point outside a circle with centre O, lines PAB and PCD are drawn to cut the circle in A, B and C, D respectively. Prove that $\triangle APD$ and $\triangle CPB$ are equiangular.
4. Two chords AB and CD of a circle intersect at P within it. If AP=AC, show that DP=DB.

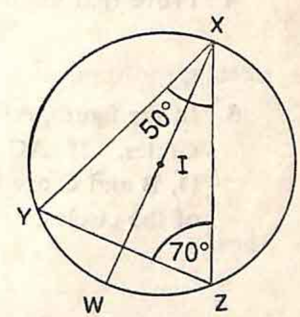
(Section B)

5. In the figure, M, A, B and N are points on a circle having centre O. AN and MB cut at Y. $\angle NYB = 50°$ and $\angle YNB = 20°$. Find the angle MAN and the reflex angle MON.



6. In the diagram I is the incentre of triangle XYZ, XI produced meets the circumcircle of triangle XYZ at W, $\angle YXZ = 50°$ and $\angle XZY = 70°$.

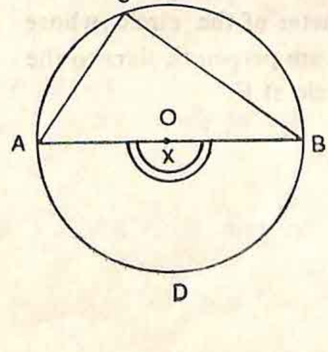
   Calculate (*a*) $\angle WYZ$
   (*b*) $\angle IYZ$
   (*c*) $\angle YIW$



7. Prove that the bisectors of the angles in the segment of a circle are congruent.

THEOREM 60

The angle in a semi-circle is a right angle.



Given : A semi-circle AOBC with centre O, and any angle ACB in the semi-circle.

To Prove : $\angle ACB = 1$ rt. $\angle$

Const. : Complete the circle.

Proof : Arc ADB subtends $\angle AOB$ at the centre O and $\angle ACB$ at a point C on the remaining part of the circumference.

$\therefore \quad \angle ACB = \frac{1}{2} \angle AOB$

or $\quad \angle ACB = \frac{1}{2} \angle x$

But $\quad \angle x = 2$ rt $\angle$s (AOB is a line, being a diameter.)

$\therefore \quad \angle ACB = \frac{1}{2} \times 2$ rt $\angle$s

or $\quad \angle ACB = 1$ rt $\angle$. Proved

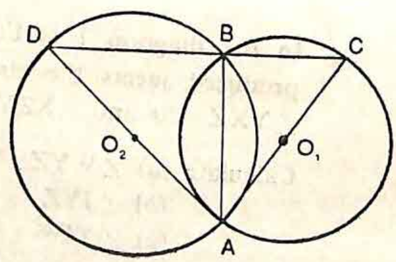
EXERCISE 7 (h)

(Section A)

1. Prove that the circles described on any two sides of a triangle as diameters intersect on the third side.
2. Two circles intersect at P and Q. Through P two diameters PA and PB are drawn onee in each circle. Prove that points A, Q, B are in the same line.
3. AB is a chord of a circle whose centre is C. Prove that the circle described on AC as diameter bisects AB.

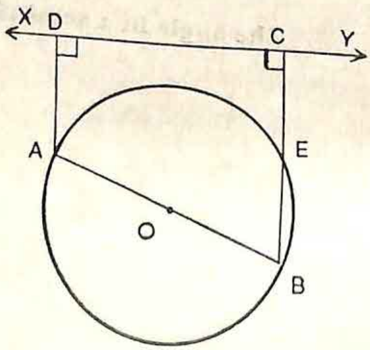
(Section B)

4. Prove that angle in a major segment is acute and that in a minor segment is obtuse.
5. In the figure, AB is the common chord of the two circles. If AC and AD are diameters, prove that D, B and C are in a line. O_1 and O_2 are the centres of the circles.



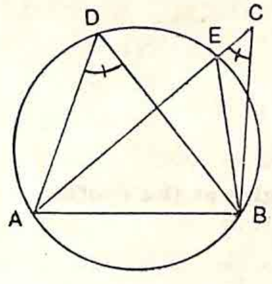
6. If the diagonals of a quadrilateral inscribed in a circle pass through the centre of th circle, prove that the quadrilateral is a rectangle.
7. In the figure, AB is a diameter of the circle whose centre is O. AD and BC are perpendiculars to the line XY. CB meets the circle at E.

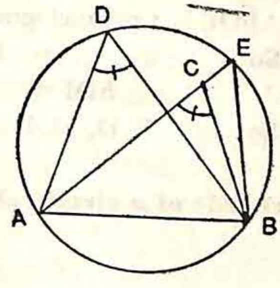
   Prove that CE=AD.



THEOREM 61

If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the segment, the four points lie on the same circle. [*C.B.S.E., 1983 (Delhi)*]





Given : Four points A, B, C and D.

Line segment AB subtends $\angle ACB$ at C and $\angle ADB$ at D on the same side of the line containing AB such that $\angle ADB = \angle ACB$.

To Prove : Points A, B, C and D are concyclic *i.e.*, lie on the same circle.

Const. : Draw a circle to pass through three non-collinear points A, B and D.

Proof : If the circle does not pass through the point C, it will intersect the line containing AC at a point E.

Now $\angle ADB$ and $\angle AEB$ are in the same segment ADEB of the circle.

$\therefore$ $\angle ADB = \angle AEB$

But $\angle ADB = \angle ACB$ (Given)

$\therefore$ $\angle AEB = \angle ACB$

This means that an exterior angle of $\triangle BCE$ is equal to an interior opposite angle,

which is impossible unless E coincides with C.

Thus, our assumption that the circle does not pass through C is false.

$\therefore$ the circle passing through A, B, D must pass through C.

i.e., the four points A, B, C, D are concyclic. Proved.

EXERCISE 7 (i)

1. Prove that four vertices of a regular pentagon are concyclic.

Hint : Let ABCDE be a regular pentagon.

Join AC and BE.

$\triangle ABE \cong \triangle ABC$

Then $\angle AEB = \angle ACB$.

2. Prove that the middle points of the sides of a triangle and the foot of the perpendicular from the vertex to the opposite side are concyclic.

Hint : D, E, F are the mid-points of the sides.

$AM \perp BC$.

Join DE, DF, ME, MF and FE.

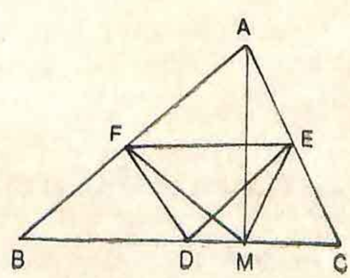
In the rt. $\angle$d $\triangle AMC$, E is the mid-point of the hyp. AC.

$\therefore$ $ME = AE = EC$

$\therefore$ $\angle MAE = \angle AME$ ...(1)

In the rt. $\angle$d $\triangle AMB$, F is the mid-point of the hyp. AB.

$\therefore$ $MF = AF = FB$



$\therefore\ \angle MAF = \angle AMF$ ...(2)

Adding (1) and (2), we have

$\angle MAE + \angle MAF = \angle AME + \angle AMF$

i.e., $\angle A = \angle EMF$

AFDE is a parallelogram.

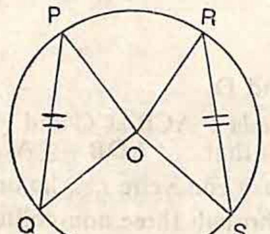
So $\angle A = \angle EDF$

$\therefore\ \angle EDF = \angle EMF$

So, F, D, M, E are concyclic.

THEOREM 62

Equal chords of a circle subtend equal angles at the centre.



Given : Chords PQ and RS of a circle with centre O such that PQ=RS

PQ subtends $\angle POQ$ and RS subtends $\angle ROS$ at the centre O.

To Prove : $\angle POQ = \angle ROS$

Proof : In $\triangle POQ$ and $\triangle ROS$

OP=OR [Radii of the same circle]

OQ=OS [Radii of the same circle]

PQ=RS [Given]

$\therefore\ \triangle POQ \cong \triangle ROS$ [SSS Congruency Theorem]

Then $\angle POQ = \angle ROS$ [Corresponding parts of congruent triangles]

Proved.

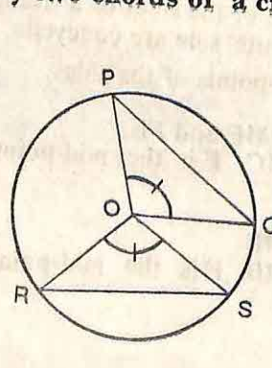
This theorem is also true for *congruent* circles. It can be stated as :

Equal chords of congruent circles subtend equal angle at the corresponding centres.

THEOREM 63

(*Converse of Theorem 62*)

If the angles subtended by two chords of a circle at the centre are equal, the chords are equal.



Given : Two chords PQ and RS of a circle with centre O.

Chord PQ subtends $\angle POQ$ and chord RS subtends $\angle ROS$ at the centre O such that $\angle POQ = \angle ROS$

To Prove : $PQ = RS$

Proof : In $\triangle POQ$ and $\triangle ROS$

$OP = OR$	[Radii of the same circle]
$OQ = OS$	[Radii of the same circle]
Incl. $\angle POQ$ = Incl. $\angle ROS$	[Given]
$\therefore \triangle POQ \cong \triangle ROS$	[SAS Congruence Axiom]
Then $PQ = RS$	[Corresponding parts of congruent triangles]

Proved.

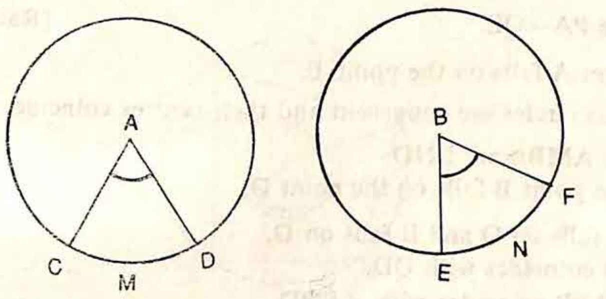
This theorem is also true for *congruent* circles. It can be stated as :

If the angles subtended by two chords of congruent circles at the corresponding centres are equal, the chords are equal.

Its proof is similar as above. Write down the proof.

THEOREM 64

In congruent circles two arcs that subtend equal angles at the corresponding centres are congruent.



Given : Two congruent circles with centres A and B and $\angle CAD = \angle EBF$.

To Prove : arc CMD $\cong$ arc ENF.

Proof : Place the circle with centre A over the circle with centre B so that A falls on B and AC falls along BE.

Since $AC = BE$, radii of congruent circles, the point C falls on the point E.

But $\angle CAD = \angle EBF$ [Given]

Then AD falls along BF.

Because $AD = BF$, radii of congruent circles, the point D falls on the point F.

The circles are congruent and their centres coincide.

$\therefore$ These circles coincide with each other.

Thus, C falls on E, D falls on F and circles also coincide.

So, arc CMD coincides with arc ENF.

Hence arc CMD $\cong$ arc ENF. Proved.

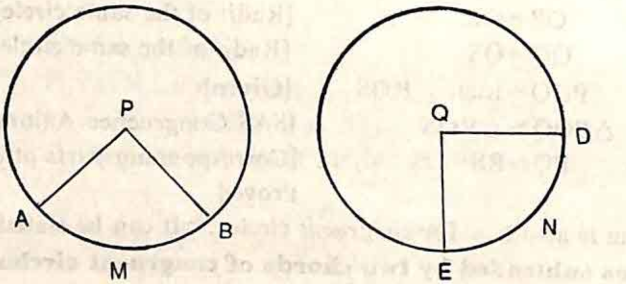
The above theorem is also true when congruent arcs are taken in the same circle. Then it is stated thus :

Two arcs of a circle are congruent, if the angle subtended by them at the centre are equal.

THEOREM 65

(*Converse of Theorem 64*)

In congruent circles, if two arcs are congruent, they subtend equal angles at the corresponding centres.



Given : Two congruent circles with centres P and Q.
Arc AMB≅Arc END.

To Prove : ∠APB=∠EQD.

Proof : Place the circle with centre P over the circle with centre Q so that P falls on Q and PA falls along QE.

Because PA=QE. [Radii of congruent circles]

the point A falls on the point E.

Since the circles are congruent and their centres coincide, they also coincides.

But arc AMB≅arc END [Given]

∴ The point B falls on the point D.

Now P falls on Q and B falls on D.

∴ PB coincides with QD.

∴ ∠APB coincides with ∠EQD.

i.e., ∠APB=∠EQD Proved.

EXERCISE 7 (j)

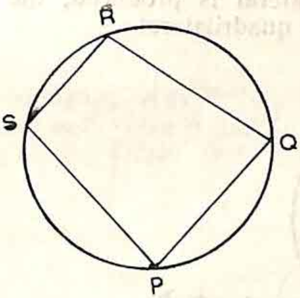
1. Prove that the bisector of an angle subtended by an arc at the centre of a circle bisects the arc.
2. Prove that the line joining the mid-point of an arc to the centre of a circle bisects the angle subtended by that arc at the centre.
3. P is a point on the circumference of a circle equidistant from radii OA and OB. Prove that arc AP≅arc BP.
4. Through O, the centre of a circle a radius. OC is drawn parallel to the chord AB. If AO produced meets the circle in D, prove that arc BC≅arc CD.

7·5. CYCLIC QUADRILATERAL

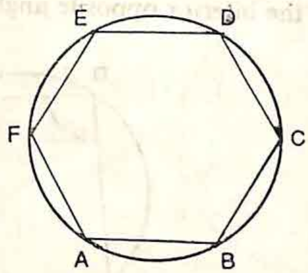
Points are said to be concyclic if they lie on the circumference of a circle.

A quadrilateral is said to be cyclic when all its vertices lie on the circumference of a circle.

Similarly a rectilinear figure is said to be cyclic when all its vertices lie on the circumference of a circle.



(*i*)



(*ii*)

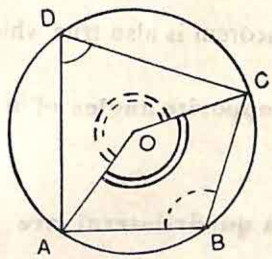
In Fig. (*i*), points P, Q, R, S are concyclic and in Fig. (*ii*) points A, B, C, D, E, F, are concyclic.

Also in Fig. (*i*), quadrilateral PQRS is cyclic and in Fig. (*ii*), ABCDEF is a cyclic hexagon.

The polygon is **inscribed** in the circle and the circle is **circumscribed** about the polygon.

THEOREM 66

The opposite angles of any quadrilateral inscribed in a circle *i.e.*, a cyclic quadrilateral are, supplementary. [*C.B.S.E., 1980 (A.I)*]



Given : A quadrilateral ABCD inscribed in a circle whose centre is O.

To Prove : $\angle A+\angle C=2$ rt. $\angle$s.

$\angle B+\angle D=2$ rt. $\angle$s.

Const. : Join AO and OC.

Proof : Arc ABC subtends $\angle AOC$ at the centre **O** and $\angle ADC$ at **a** point **D** on the remaining part of the circumference.

$\therefore \quad \angle ADC=\frac{1}{2}\angle AOC$ ...(*i*)

Arc ADC subtends reflex $\angle AOC$ at the centre and $\angle ABC$ on the circumference.

$\therefore \quad \angle ABC=\frac{1}{2}$ reflex $\angle AOC$ ...(*ii*)

Adding (*i*) and (*ii*), we have

$\angle ADC+\angle ABC=\frac{1}{2}[\angle AOC+\text{reflex }\angle AOC]$

$=\frac{1}{2}\times 4$ rt. $\angle$s [Angles around the point O]

$=2$ rt. $\angle$s

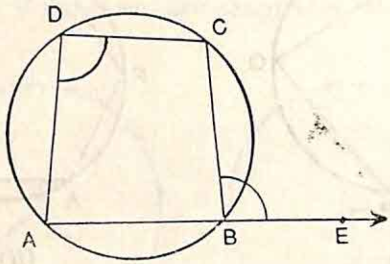
i.e., $\angle D+\angle B=2$ rt. $\angle$s

Similarly

$\angle A+\angle C=2$ rt. $\angle$s

Proved.

Corollary : If one side of a cyclic quadrilateral is produced, the exterior angle so formed is equal to the interior opposite angle of the quadrilateral.



Given : A cyclic quadrilateral ABCD whose side AB is produced to E.

To Prove : Ext. $\angle CBE$=Int. opp. $\angle ADC$.

Proof : $\angle ABC+\angle CBE=180°$ ...(1) [Adjacent angles on a line ABE.]

$\angle ABC+\angle ADC=180°$ ...(2) [Opposite angles of a cyclic quadrilateral ABCD]

From (1) and (2), we have

$\angle ABC+\angle CBE=\angle ABC+\angle ADC$

$\therefore \quad \angle CBE=\angle ADC$

Proved.

The converse of the above theorem is also true which can be stated without proof as under :

If the sum of any pair of opposite angles of a quadrilateral is 180°, then the quadrilateral is cyclic.

Or

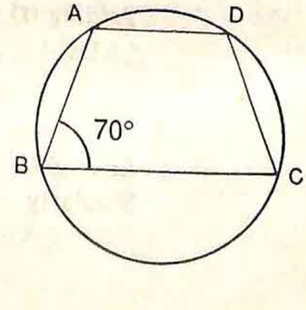
If the opposite angles of a quadrilateral are supplementary, its vertices are concyclic.

EXERCISE 7 (k)

(Section A)

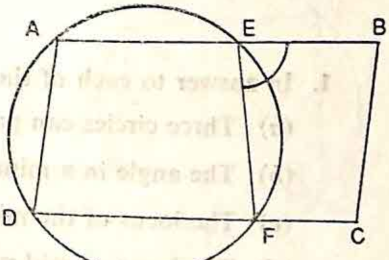
1. Prove that a parallelogram inscribed in a circle is a rectangle. [*C.B.S.E., 1979 (Delhi)*]
2. If the opposite sides of a cyclic quadrilateral are equal, prove that it is a rectangle.
3. If two sides of a cyclic quadrilateral are parallel, show that the other two sides are equal.
4. Two right-angled $\triangle$s ACB and ADB are on opposite sides of a common hypotenuse AB. If CD is joined, prove that $\angle BAD=\angle BCD$.
5. PQRS is a cyclic quadrilateral in which $\angle SPQ=85°$, $\angle SQP=58°$, calculate $\angle PRQ$.
6. Given a cyclic trapezium ABCD in which AD is parallel to BC and $\angle B=70°$ as shown in the adjoining figure.

   Find $\angle BAD$ and $\angle BCD$.

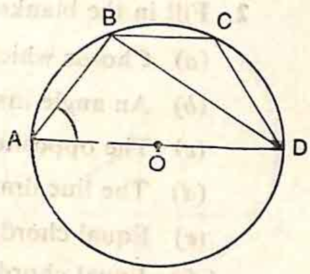


(Section B)

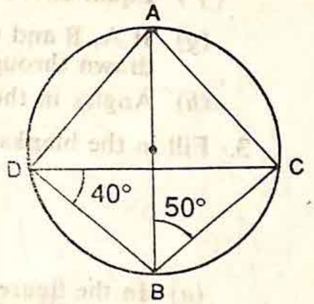
7. In the given figure, ABCD is a parallelogram. A circle passes through A and D cuts AB at E and DC at F. Given that $\angle BEF = 80°$, find $\angle ABC$.



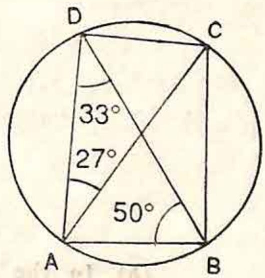
8. In the adjoining figure ABCD, if $\angle BCD = 125°$ and AD is the diameter of the circle, calculate
   (*i*) $\angle DAB$
   (*ii*) $\angle ADB$.



9. In the figure given alongside, if $\angle ABC = 50°$ and $\angle BDC = 40°$, calculate
   (*a*) $\angle CDA$,
   (*b*) $\angle BAC$,
   (*c*) $\angle BCA$.



10. In the given figure, ABCD is a cyclic quadrilateral in which $\angle DAC = 27°$, $\angle DBA = 50°$ and $\angle ADB = 33°$, calculate
   (*a*) $\angle DBC$,
   (*b*) $\angle DCB$,
   (*c*) $\angle CAB$.



11. A triangle is inscribed in a circle. Prove that the sum of the angles in the three segments exterior to the triangle is equal to four right angles.
12. In a cyclic hexagon ABCDEF, AB ∥ ED and CD ∥ AF ; prove that BC ∥ FE.
    Hint : Join AD and CF.

(Section C)

13. ABCD is a parallelogram. A circle is drawn through A and B intersects AD and BC (produced if necessary) in E and F respectively. Prove that the points C, D, E, F are concyclic.
14. Two circles intersect at P and Q. Through P a line AB is drawn to meet the circles in A and B. Through Q a line CD is drawn to meet the circles in C and D. Prove that AC is parallel to BD.
15. Prove that the altitudes of a triangle are concurrent.

REVIEW EXERCISE VI

(Section A)

1. In answer to each of the following statements write True or False :

(*a*) Three circles can pass through three given points not in a line.

(*b*) The angle in a minor segment of a circle is an obtuse angle.

(*c*) The locus of the mid-points of all equal chords of a circle is a circle.

(*d*) The locus of mid-points of the radii of a circle is a circle.

2. Fill in the blanks to make each of the following statements true :

(*a*) Chords which are equidistant from the centre of a circle are.........

(*b*) An angle inscribed in a semi-circle is a......... [*C.B.S.E., 1984 (A.I.)*]

(*c*) The opposite angles of a cyclic quadrilateral are............

(*d*) The line drawn through the centre to bisect a chord is...........

(*e*) Equal chords of a circle are...........from the centre. [*C.B.S.E., 1985 (A.I.)*]

(*f*) Equal chords of a circle subtend...........at the centre. [*C.B.S.E., 1982 (A.I.)*]

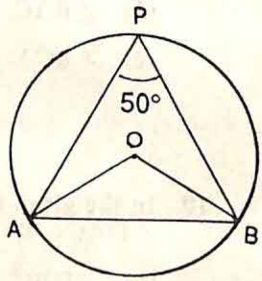
(*g*) If A, B and C are three non-collinear points, then exactly...........circle(s) can be drawn through all of these points. [*C.B.S.E., 1980 (A.I.)*]

(*h*) Angles in the same segment of a circle are........... [*C.B.S.E., 1979 (A.I.)*]

3. Fill in the blanks making each of the following a true statement :

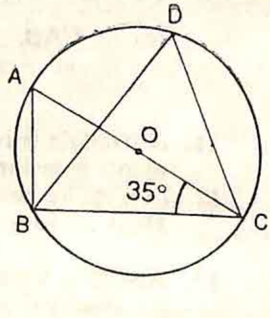
(*a*) In the figure, O is the centre of the circle. If $\angle APB = 50°$, then $\angle OAB =$...........

[*C.B.S.E., 1982 (Delhi)*]



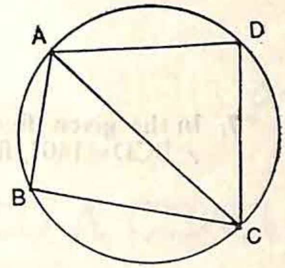
(*b*) In the figure, if O is the centre of the circle and $\angle ACB = 35°$ then $\angle BDC =$...........

[*C.B.S.E., 1984 (A.I.)*]



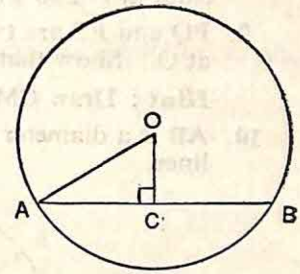
(c) In the figure, if $\angle BAC = 60°$ and $\angle BCA = 20°$, then $\angle ADC = $.........

[*C.B.S.E., 1983 (A.I.)*]



(d) In the figure, if OC is perpendicular to AB, OA=5 cm and OC=3 cm, then AB=.........

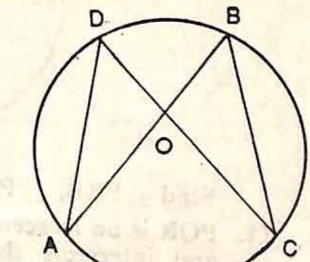
[*C.B.S.E., 1981 (Delhi)*]



4. Two chords AB and CD of a circle intersect each other at the point O inside the circle.

Prove that $\triangle AOD \sim \triangle COB$.

[*C.B.S.E., 1986 (A.I.)*]

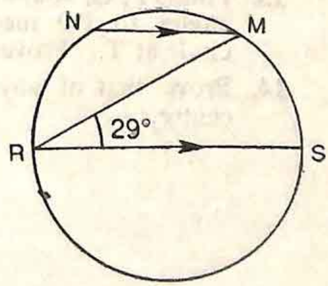


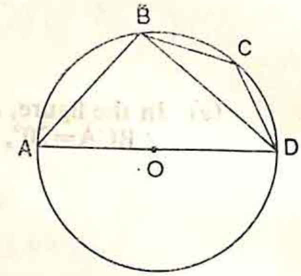
5. AB and CD are parallel chords of a circle whose diameter is AC. Prove that AB=CD.
[*C.B.S.E., 1980 (Delhi); 1983 (A.I.)*]

(Section B)

6. In the given figure, RS is a diameter of a circle. NM is parallel to RS and $\angle MRS = 29°$.

Calculate $\angle RNM$ and $\angle NRM$.





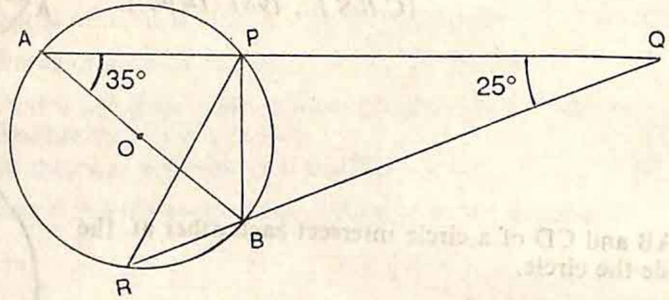
7. In the given figure, AB is a diameter of the circle. If $\angle BCD=140°$, find $\angle DBA$.

8. ABC is an isosceles triangle and a line DE is drawn parallel to the base intersecting the sides in D and E. Prove that B, C, E and D lie on a circle. [*C.B.S.E., 1984 (Delhi)*]

9. PQ and RS are two equal chords of a circle with centre C. They are produced to meet at O. Show that OQ=OS. [*C.B.S.E., 1982 (A.I.) ; 1984 (A.I.)*]

Hint : Draw $CM \perp PQ$, $CN \perp RS$ and join CO.

10. AB is a diameter of the circle APBR as shown in the figure. APQ and RBQ are two lines.



Find $\angle PRB$, $\angle PBR$ and $\angle BPR$.

11. PQR is an isosceles triangle with PQ equal to PR. A circle passes through Q and R and intersects the sides PQ and PR at points S and T respectively. Prove that QR is parallel to ST.

12. Chords AB and CD intersect at right angles at a point inside the circle, and $\angle BAC=40°$.
    (*i*) Sketch the chords AB and CD, and mark the angle BAC, in the circle.
    (*ii*) From the figure, calculate the value of $\angle ABD$.

(Section C)

13. Points P, Q, R are taken on the circumference of a circle such that PS drawn at right-angles to PQ meets the circle at S ; and RT drawn at right-angles to PR meets the circle at T. Prove that PQ=ST.

14. Prove that of any two chords of a circle, the one which is greater is nearer the centre.

□□

8

TANGENT TO A CIRCLE

8·1. SECANT AND TANGENT

If a circle and a line are drawn on a plane, there are *three different situations* as shown below :

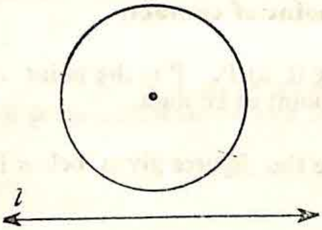


Fig. 1

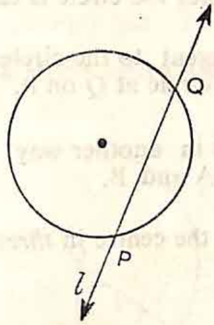


Fig. 2

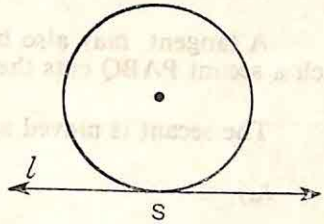


Fig. 3

In Fig. 1, line l *does not* intersect the circle.

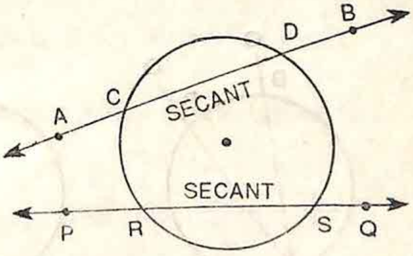
In Fig. 2, line l intersects the circle in *two distinct points* P and Q. Then we say that line l is a **secant**.

In Fig. 3, line l intersects the circle in *only one point* S.

Then we say that line l is a **tangent**.

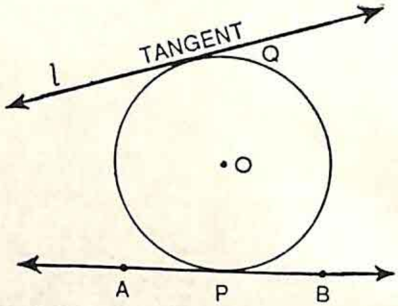
A line which intersects a circle in two distinct points is called a secant of the circle.

Here lines AB and PQ are two secants of the circle. Secant AB intersects the circle at C and D. Secant PQ intersects the circle at R and S.



A tangent to a circle is a line that intersects the circle in exactly one point.

Here line *l* and line AB are two tangents to the circle with centre O.



Since a tangent meets the circle at one and only one point, the tangent is said to *touch* the circle.

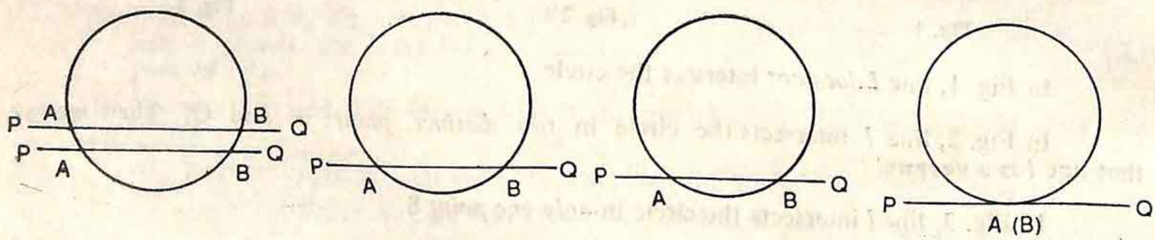
The point where a tangent touches the circle is called **the point of contact.**

Thus, in the figure, AB is a tangent to the circle, touching it at P. P is the point of contact. Also line *l* is a tangent to the circle at Q on it. Q is the point of contact.

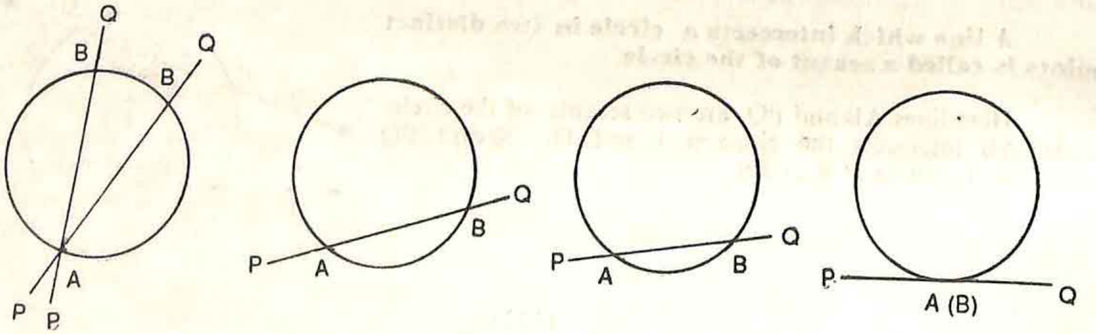
A tangent may also be defined in another way. Examine the figures given below in which a secant PABQ cuts the circle at A and B.

The secant is moved away from the centre in *three* ways :

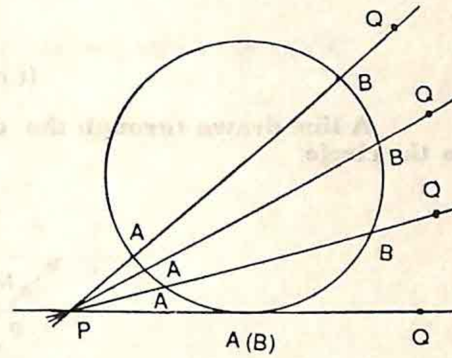
(*a*)



(*b*)



(c)



(*i*) parallel to itself as shown in figures (*a*) ;

(*ii*) about the point A as shown in figure (*b*) ; and

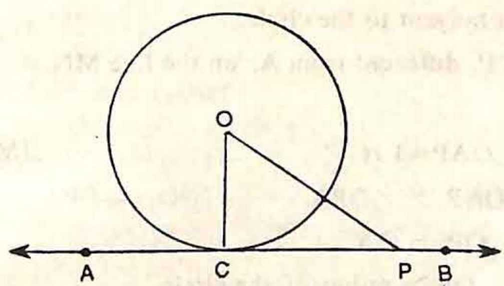
(*iii*) about the point P as shown in figure (*c*).

In all the cases, the points of intersection, A and B come *nearer* to each other and at one stage they coincide. Then the secant becomes a tangent to the circle.

Hence **a tangent to a circle is the limiting position of a secant when the points of intersection with the circle coincide.**

THEOREM 67

The tangent at any point of a circle and the radius through the point of contact are perpendicular to one another. [*C.B.S.E., 1985 (A.I.)*]



Given : A circle with centre O.
Line ACB is a tangent at C.
Line segment OC is the radius through the point of contact C.

To Prove : OC $\perp$ ACB.

Const. : Take any point P in AB.
Join OP.

Proof : Line AB is a tangent to the circle at C.

$\therefore$ Every point in line AB except the point of contact C lies outside the circle.

Then point P lies outside the circle.

Since the distance of an outside point from the centre is greater than the radius of the circle,

radius OC $<$ OP.

Similarly it can be proved that OC is less than any other line segment drawn from O to AB.

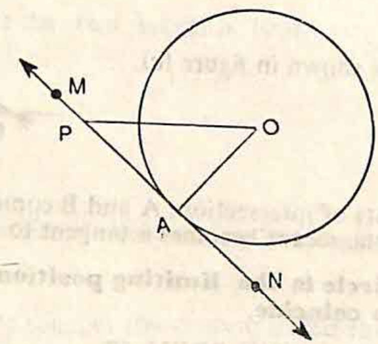
$\therefore$ OC is the shortest of all the line segments that can be drawn from the centre O to the tangent AB.

Hence OC $\perp$ ACB. Proved.

THEOREM 68

(*Converse of Theorem 67*)

A line drawn through the end of a radius and perpendicular to it is a tangent to the circle.



Given : A radius OA of a circle (O, r).

Line MN through A and perpendicular to OA.

To Prove : Line MN is a tangent to the circle.

Const. : Take a point P, different from A, on the line MN.

Join OP.

Proof : In $\triangle AOP$, $\angle OAP = 1$ rt. $\angle$ [MN $\perp$ OA, given]

$\therefore$ $\angle OAP > \angle OPA$

$\therefore$ $OP > OA$

i.e., OP $>$ radius of the circle.

$\therefore$ Point P lies outside the circle.

Thus, every point on the line MN except A lies outside the circle.

$\therefore$ the line MN meets the circle at only one point.

Hence the line MN is a tangent to the circle. Proved.

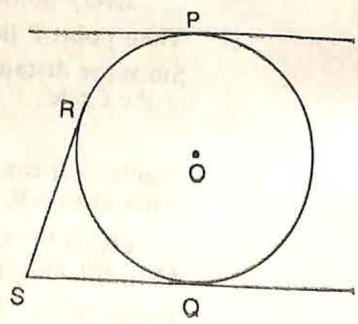
8·2. TANGENT-SEGMENTS

Tangents drawn at two points of a circle may be either *parallel or intersecting*.

In the adjoining figure, tangents at the points P and Q to the circle are parallel, while the tangents at the points Q and R intersect in a point S. We can also say that both the tangents at Q and R pass through the point S.

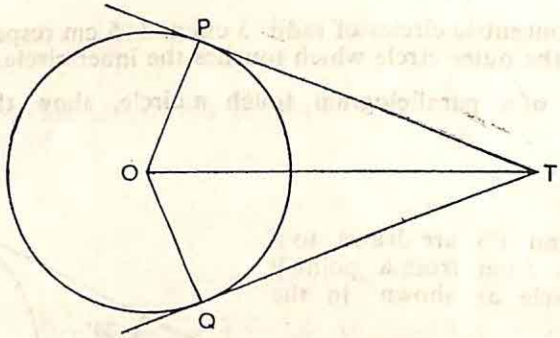
We can observe from the figure that no other tangent to the circle can be drawn to pass through the point S.

The line segments SQ and SR are called the **tangent-segments** and their lengths as length of tangents from S to the circle.



THEOREM 69

The length of two tangents from an external point to a circle are equal.



Given : A circle C (O, r) and an external point T.
TP and TQ are two tangent segments from T to the circle C (O, r).

To Prove : TP=TQ

Const. : Draw line segments OP, OQ and OT.

Proof : TP is a tangent segment to the circle C (O, r) and OP is the radius through the point of contact P.

$\therefore \quad \angle OPT = 90^\circ$

Similarly $\quad \angle OQT = 90^\circ$

In the right $\triangle OPT$ and $\triangle OQT$

$OP = OQ = r$ (Radii)

hyp. OT=hyp. OT (Common)

$\therefore \quad \triangle OPT \cong \triangle OQT$ (RHS-Congruency Theorem)

$\therefore \quad TP = TQ$ (Corresponding parts of congruent triangles)

Proved.

In the proof of the above theorem we observe that

$\angle TOP = \angle TOQ$ and $\angle PTO = \angle QTO$.

We can state these results as under :

If two tangents are drawn to a circle from a point outside the circle, then

(*i*) they subtend equal angles at the centre, and

(*ii*) they are equally inclined to the line segment joining the centre and the given point.

EXERCISE 8 (a)

(Section A)

1. Prove that the tangents at the extremities of a diameter of a circle are parallel. [*C.B.S.E., 1986 (A.I.)*]
2. Prove that the line segment joining the points of contact of two parallel tangents to a circle passes through the centre.
3. Prove that a perpendicular to the tangent at the point of contact passes through the centre of the circle.
4. Prove that the tangents at the extremities of a chord of a circle make equal angles with the chord. [*C.B.S.E., 1984 (Delhi)*]
5. If a quadrilateral is described about a circle, show that the sum of one pair of opposite sides is equal to the sum of the other pair.

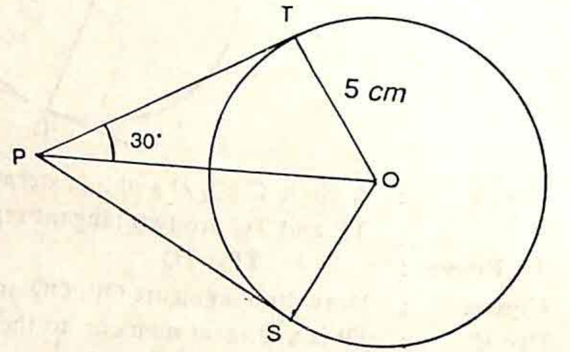
(Section B)

6. Find the length of the tangent drawn to a circle of radius 3 cm from a point distant 5 cm from the centre.

7. There are two concentric circles of radii 3 cm and 5 cm respectively. Find the length of the chord of the outer circle which touches the inner circle.

8. If all the sides of a parallelogram touch a circle, show that the parallelogram is a rhombus.

9. Tangents PT and PS are drawn to a circle of radius 5 cm from a point P outside the circle as shown in the adjoining figure.
If O is the centre of the circle and $\angle TPO = 30°$, calculate
(*a*) PO,
(*b*) PT, correct to one decimal place,
(*c*) the area of PTOS in cm², correct to one decimal place.



10. Prove that only two tangents can be drawn to a circle from an external point.

Hint : Let P be the external point.

Join OP and draw a circle on OP as a diameter.

(Section C)

11. If a chord AB of a circle is parallel to the tangent at point P on it, prove that AP=BP.

Hint. Join OP, OA and OB.

$\triangle AOC \cong \triangle BOC$

12. A tangent CD to a circle at the point E on it meets two parallel tangents in point C and D. Prove that line segment CD subtends a right angle at the centre O.

Hint. Join OA, OB and OE.

$\triangle AOC \cong \triangle EOC$

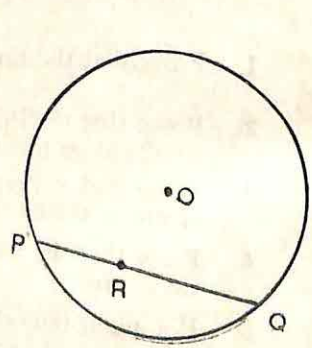
Also $\triangle BOD \cong \triangle EOD$

8·3. SEGMENTS OF A CHORD

Let PQ be a chord of a circle C(O, *r*), and R be a point on line segment PQ.

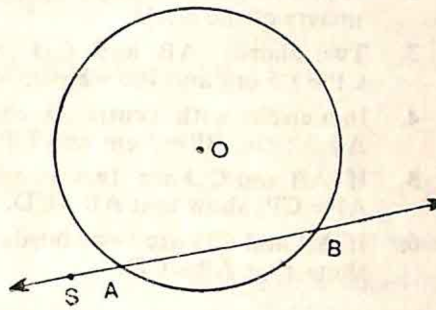
Observe that the point R lies *inside* the circle.

Then R is said to divide chord PQ **internally** into two segments PR and RQ.



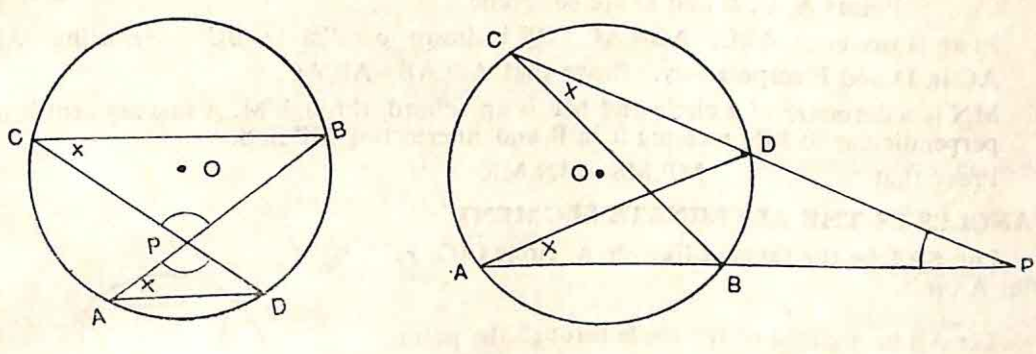
Let AB be a chord of a circle C(O, *r*) and S be a point on line PQ *outside* the circle.

Then S is said to divide chord AB **externally** into two segments SA and SB.



THEOREM 70

If two chords of a circle intersect inside or outside of the circle when produced, the rectangle formed by the two segments of one chord is equal in area to the rectangle formed by the two segments of the other.



(*i*) (*ii*)

Given : Two chords AB and CD of a circle C(O, *r*) intersecting inside or outside, when produced, of the circle at P.

To Prove : PA . PB = PC . PD.

Const. : Join AD and BC.

Proof : $\angle BAD = \angle BCD$ [Angles in the same segment]

i.e., $\angle PAD = \angle PCB$

In $\triangle APD$ and $\triangle CPB$

$\angle PAD = \angle PCB$ [Proved above]

$\angle APD = \angle CPB$ [Vertically opposite angles in figure (*i*) and common angle in figure (*ii*)]

$\therefore$ the remaining angles are equal.

So, $\triangle APD$ and $\triangle CPB$ are equiangular and hence they are similar.

$$\therefore \quad \frac{PA}{PC} = \frac{PD}{PB}$$

i.e., PA . PB = PC . PD Proved.

EXERCISE 8 (b)

(Section A)

1. M is the mid-point of a chord AB ; CD is another chord through M ; prove that CM . MD = AM^2.

2. If two chords of a circle bisect each other inside the circle, prove that they are diameters of the circle.
3. Two chords AB and CD intersect at P inside a circle such that AP=3 cm, CP=1·5 cm and PD=8 cm. Find the length of PB.
4. In a circle with centre O, chords AB and CD intersect at P outside it such that AB=2 cm, BP=4 cm and DP=3 cm. Find the length of the chord CD.
5. If AB and CD are two chords intersecting at a point P inside the circle such that AP=CP, show that AB=CD.
6. If AB and CD are two chords which when produced meet at a point P and if AP=CP, show that AB=CD.

(Section B)

7. Through P, the point of intersection of two circles, two line segments APB and CPD are drawn, each passing through a centre of a circle and meeting the other circle. Prove that AP . PB=CP . PD.

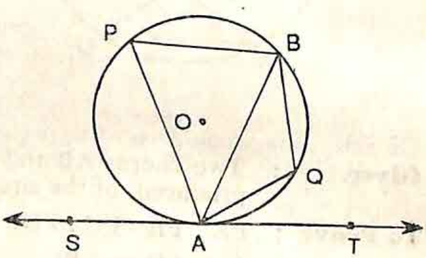
   Hint. Join AC, and BD.

   Points A, C, B and D are concyclic.
8. In an isosceles $\triangle$ ABC, AB=AC. DE is drawn parallel to BC intersecting AB and AC in D and E respectively. Prove that AD.AB=AE.AC.
9. MN is a diameter of a circle and MP is any chord through M. A line segment is drawn perpendicular to MN meeting it in R and intersecting MP in S.

   Prove that MP.MS=MN.MR

8·4. ANGLES IN THE ALTERNATE SEGMENT

Let SAT be the tangent line to a circle C(O, *r*) at point A on it.



Let AB be a chord of the circle through the point of contact A.

Chord AB makes two angles—$\angle$BAT and $\angle$BAS with the tangent line SAT.

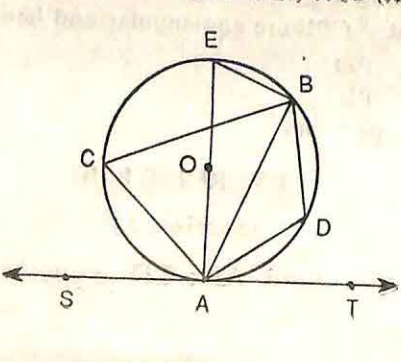
Let P and Q be any two points on the circle on either side of AB.

Then $\angle$APB is said to be an angle in the **alternate segment** of $\angle$BAT. Similarly, $\angle$AQB is an angle in the **alternate segment** of $\angle$BAS.

THEOREM 71

If a chord is drawn through the point of contact of a tangent to a circle, then the angles which this chord makes with the given tangent are equal respectively to the angles formed in the corresponding alternate segments.

[*C.B.S.E., 1982* (*A.I.*); *1984* (*A.I.*) ; *1980* (*Delhi*)]



Given : ST is a tangent line to a circle C(O, r) with point of contact A.
AB is a chord through the point of contact A.
C and D are points on the circle on the opposite sides of AB.

To Prove : $\angle BAT = \angle ACB$
and $\angle BAS = \angle ADB$

Const. : Draw the diameter AOE and join BE.

Proof. : AOE is a diameter of the circle. [Construction]

$\therefore \quad \angle ABE = 90°$ [Angle in a semi-circle]

Now in the right $\triangle ABE$,

$\angle AEB + \angle EAB = 90°$ ...(1)

ST is a tangent line to the circle at A and AOE is a diameter through the point of contact A.

$\therefore \quad \angle EAT = 90°$

i.e., $\angle EAB + \angle BAT = 90°$ ...(2)

From (2) and (1), we get

$\angle EAB + \angle BAT = \angle AEB + \angle EAB$

$\therefore \quad \angle BAT = \angle AEB$

$\angle AEB = \angle ACB$ [Angles in the same segment]

$\therefore \quad \angle BAT = \angle ACB$

Now ADBC is a cyclic quadrilateral.

$\therefore \quad \angle ACB + \angle BDA = 180°$ [Opposite angles]

i.e., $\angle BAT + \angle BDA = 180°$ ...(3) [$\angle ACB = \angle BAT$ proved.]

Line segment AE stands on the line ST at A.

$\therefore \quad \angle BAS + \angle BAT = 180°$ ...(4)

from (4) and (3), we get

$\angle BAS + \angle BAT = \angle BAT + \angle BDA$

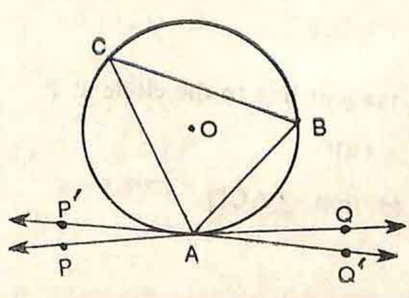
$\therefore \quad \angle BAS = \angle BDA$

i.e., $\angle BAS = \angle ADB$ Proved.

THEOREM 72

(*Converse of Theorem 71*)

If a line is drawn through an end point of a chord of a circle such that the angle which it makes with the chord is equal to the angle in the alternate segment, then the line is a tangent to the circle.



Given : AB is a chord of a circle C (O, r).

A line PAQ is drawn through A such that $\angle BAQ = \angle ACB$ in the alternate segment.

To Prove : PAQ is a tangent line to the circle at the point A.

Const. : If PAQ is not a tangent, draw tangent P'AQ' to the circle at A.

Proof : Now P'AQ' is tangent line to the circle at A and AB is a chord through the point of contact A.

$\therefore$ $\angle BAQ' = \angle ACB$ [Angle in the alternate segment]

But $\angle BAQ = \angle ACB$ [Given]

$\therefore$ $\angle BAQ = \angle BAQ'$

This is impossible unless ray AQ' coincides with ray AQ.

Therefore P'AQ' coincides with PAQ or PAQ is the tangent to the circle at A.

Proved.

EXERCISE 8 (c)

(Section A)

1. A $\triangle ABC$ in which $AB = BC$, is inscribed in a circle. Prove that the tangent at B is parallel to the chord AC.

2. DE is a tangent line to a circumcircle of $\triangle ABC$ at A such that $DE \parallel BC$. Prove that $AB = AC$. [*C.B.S.E., 1981* (*A.I.*) ; *1982* (*Delhi*)]

3. A line segment PAB intersects a circle in A and B, and PC is a tangent segment to the circle. If $\angle PCB = 115°$, find the $\angle BAC$.

4. The chord AB of a circle is produced to C and CT is a tangent segment to the circle. If $BC = BT$ and $\angle CBT = 114°$, find the angles of the $\triangle ATB$.

(Section B)

5. A tangent to a circle is drawn parallel to a chord of the circle. Prove that the point of contact bisects the arc cut off by the chord.

6. Two circles intersect at A and B. From a point P on one of these circles two line segments PAC and PBD are drawn intersecting the other circle at C and D respectively. Prove that CD is parallel to tangent at P.

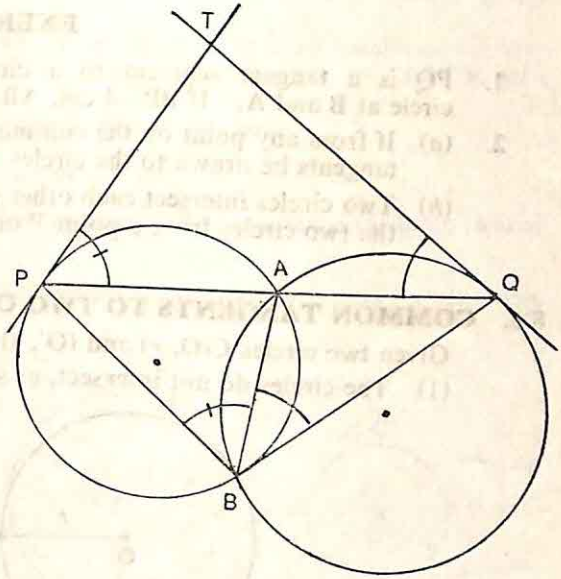
Hint : Join AB.

Let XPY be a tangent line to the circle at P.

$\angle APX = \angle ABP$

ext. $\angle ABP$ = int. opp. $\angle ACD$

(Section C)



7. Two circles intersect at A and B, a line segment PAQ intersects the circles at P and Q. If the tangents at P and Q intersect at T, prove that the points P, B, Q, T are concyclic.

Hint : Join AB, BP and BQ.

$\angle APT = \angle ABP$

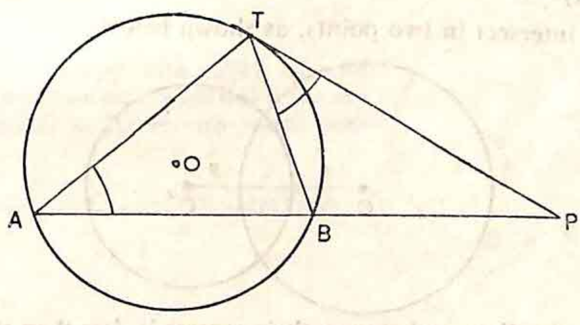
$\angle AQT = \angle ABQ$

$\angle APT + \angle AQT = \angle PBQ$

Then $\angle PBQ + \angle T = \angle APT + \angle AQT + \angle T = 2$ rt. $\angle s$.

THEOREM 73

If PAB is a secant to a circle intersecting the circle at A and B and PT is a tangent segment, then PA . PB$=$PT^2 [*C.B.S.E.*, *1983* (*A.I.*)]



Given : A secant ABP to a circle C (O, r) intersecting it in A and B. PT is a tangent segment to the circle from P.

To Prove : PA . PB$=$PT^2.

Const. : Join AT and BT.

Proof : PT is a tangent segment to the circle at T and TB is a chord through the point of contact T.

$\angle PTB = \angle BAT$ [Angle in the alternate segment]

i.e., $\angle PTB = \angle PAT$

Now in $\triangle PBT$ and $\triangle PTA$

$\angle PTB = \angle PAT$ [Proved above]

$\angle BPT = \angle TPA$ [Common angle]

$\therefore$ the remaining angles are equal.

So, $\triangle PBT$ and $\triangle PTA$ are equiangular and hence they are similar.

$$\therefore \qquad \frac{PB}{PT} = \frac{PT}{PA}$$

i.e., $\qquad PA \cdot PB = PT^2 \qquad$ Proved.

EXERCISE 8 (d)

1. PQ is a tangent segment to a circle at point Q and PBA is a secant intersecting the circle at B and A. If BP=4 cm, AB=12 cm, find the length of PQ.

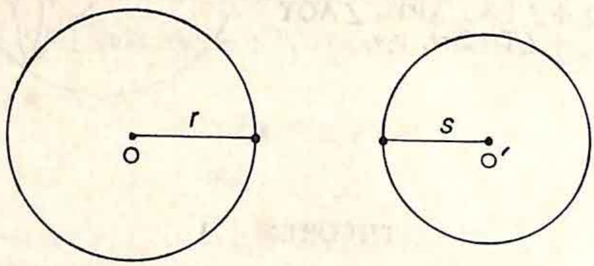
2. (*a*) If from any point on the common chord produced, of two intersecting circles, tangents be drawn to the circles, prove that they are equal.

(*b*) Two circles intersect each other in points A and B. If PS and PT are tangents to the two circles from a point P on the line containing A and B, show that PS=PT. [*C.B.S.E., 1985 (A.I.)*]

8·5. COMMON TANGENTS TO TWO CIRCLES

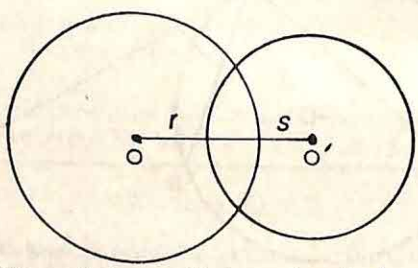
Given two circles C(O, r) and (O′, s). There are three possibilities.

(1) The circles do not intersect, as shown below :



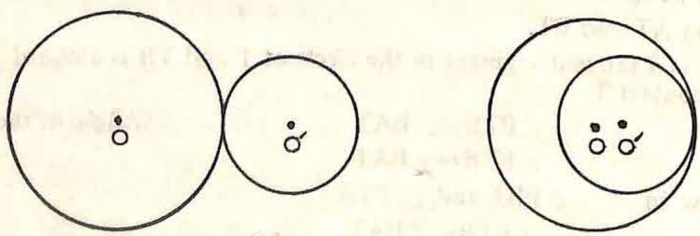
We observe that the distance between their centres is *greater* than the sum of the radii of the circles. So, $OO' > r+s$.

(2) The circles intersect in two points, as shown below :



We observe that the distance between their centres is *less* than the sum of the radii of the circles. So, $OO' < r+s$.

(3) The circles intersect in only one point, as shown below :



(i) (ii)

When two circles intersect in only one point, they are said **to touch** each other.

Two circles can touch each other in *two ways*. See Figs. (*i*) and (*ii*) above.

When each of the touching circles lies outside the other, they are said to touch one another **externally**. [See Fig. (*i*)]

When one of the touching circles is within the other, they are said to touch one another **internally**. [See Fig. (*ii*)]

The point at which the circles meet is called **the point of contact.**

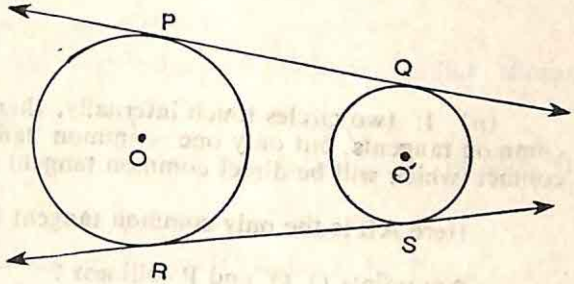
The line joining their centres is called the **line of centres.**

If a line touches each of two given circles, it is called a common tangent to the circles.

If two circles do not intersect, *two pairs of common tangents* can be drawn to the two circles.

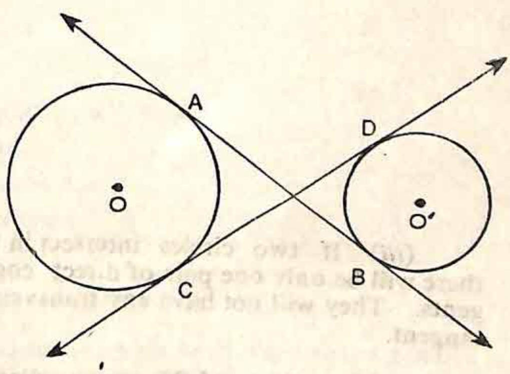
If the circles lie on the same side of the common tangent, the tangent is called a direct common tangent.

In the figure, PQ and RS are two direct common tangent lines.



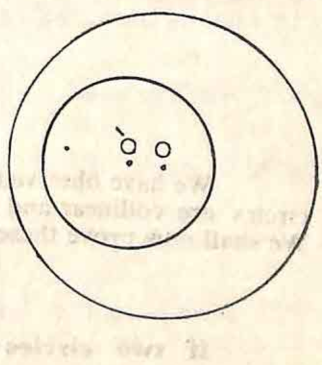
If the circles lie on opposite sides of the common tangent, the tangent is called a transverse common tangent or indirect common tangent.

In the figure, AB and CD are two transverse common tangent lines.



When one circle is inside the other circle completely, *no* common tangent can be drawn to the two circles.

Note that common tangents to two circles will exist if one circle does not lie inside the other.

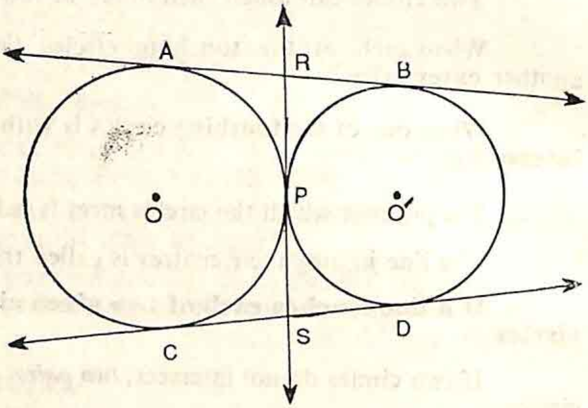


Let us now discuss some other cases.

(*i*) If two circles touch externally, there will a pair of direct common tangents and also a common tangent at the point of contact of the circles which will be the transverse common tangent.

Here lines AB and CD are direct common tangents and line RS is a transverse common tangent.

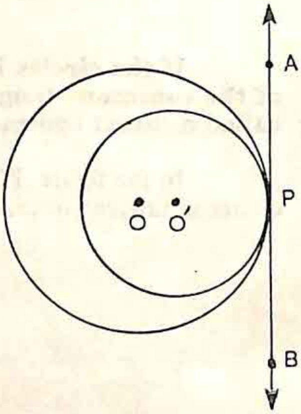
Are points O, O′ and P collinear ?



(*ii*) If two circles touch internally, there will be no transverse common tangents, but only one common tangent at the point of contact which will be direct common tangent to the two circles.

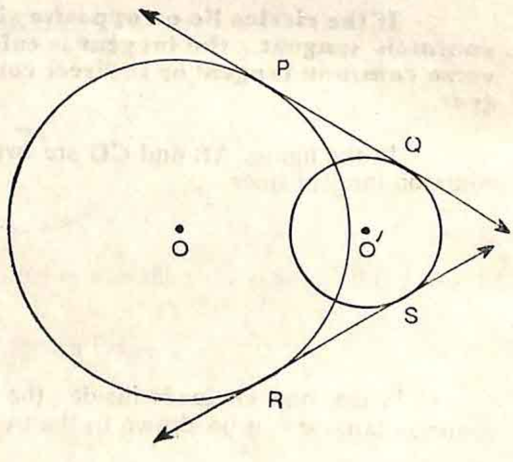
Here AB is the only common tangent line to the two circles.

Are points O, O′ and P collinear ?



(*iii*) If two circles intersect in two points, there will be only one pair of direct common tangents. They will not have any transverse common tangent.

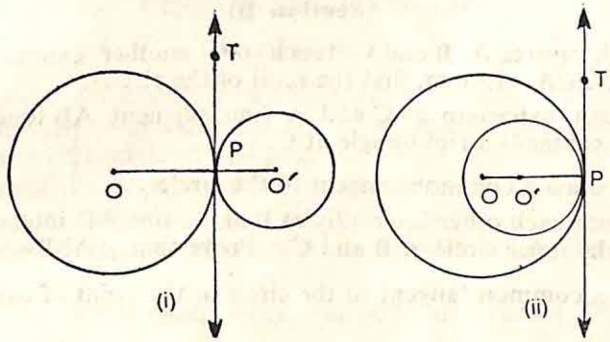
Here PQ and RS are two direct common tangents.



We have observed that the two centres and the point of contact of the two touching circles are collinear and there will be a common tangent to these circles at the point of contact. We shall now prove these facts.

THEOREM 74

If two circles touch each other, the point of contact lies on the line through the centres. [*C.B.S.E., 1980 (A.I.); 1985 (A.I.); 1984 (Delhi); 1985 (Delhi)*]



Given : Two circles with centres O and O′ touching each other at P externally [Fig. (*i*)] or internally [Fig. (*ii*)].

To Prove : Points O, P and O′ lie on the same line.

Const. : Join OP and PO′.

Draw a common tangent PT to the circles at P.

Proof : For the circle with centre O, PT is a tangent line and PO is the radius through the point of contact.

$\therefore$ $\angle OPT = 90°$ *i.e.,* $PO \perp PT$.

For the circle with centre O′, PT is a tangent and PO′ is the radius through the point of contact P.

$\therefore$ $\angle O'PT = 90°$ *i.e.,* $PO' \perp PT$.

Thus, PO and PO′ are both perpendicular to line PT at the point P.

But through a given point one and only one perpendicular can be drawn to the given line.

Hence PO and PO′ must be the same line.

So, O, P and O′ lie on the same line. Proved.

Corollary : The two circles C (O, r) and C (O′, s) will touch

(*i*) externally, if and only if, $OO' = r + s$, and

(*ii*) internally, if and only if, $OO' = r - s$, when $r > s$

$OO' = s - r$, when $s > r$.

EXERCISE 8 (e)

(Section A)

1. Two circles of radii 1·8 cm and 24 mm are drawn touching each other (*a*) externally, or (*b*) internally. Find the distance between their centres in each case.

2. Three circles of radii 10 mm, 15 mm and 20 mm respectively are drawn so that each touches the other two externally. Find the lengths of the sides of the triangle whose vertices are the centres of the circles.

3. Three circles with equal radii touch each other externally. Show that the triangle formed by joining their centres is equilateral.

4. Prove that the tangents at the point of contact of two circles touching each other externally bisects their direct common tangents.

5. Two circles touch at P. Through P a line segment is drawn cutting the circle at M and N. Prove that the radii through M and N are parallel.

6. Two circles touch each other externally at P and through P a line segment APB is drawn cutting the circles at A and B. Prove that the tangents at A and B are parallel.

(Section B)

7. The circles with centres A, B and C touch one another externally. If AB=17 mm, BC=22 mm and CA=19 mm, find the radii of the circles.

8. Two circles touch externally at C and a line segment AB touches them at A and B. Prove that AB subtends a right-angle at C.

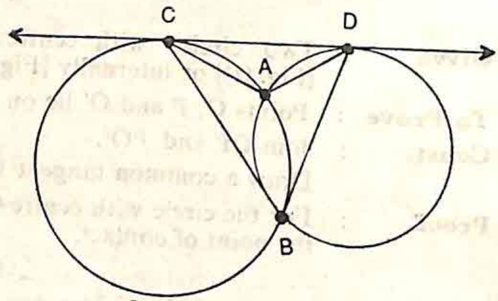
   Hint : At C, draw a common tangent to the circles.

9. Two circles touch each other internally at P and a line AD intersects the outer circle at A and D and the inner circle at B and C. Prove that $\angle APB = \angle CPD$.

   Hint : Draw a common tangent to the circle at the point of contact P.

10. Two circles intersect each other at the points A and B. CD is a direct common tangent as shown in the adjoining figure. Prove that the angles subtended by the segment CD at A and B are supplementary.

    [*C.B.S.E., 1984 (A.I.)*]



(Section C)

11. Two circles touch each other at P. APC and BPD are lines through P that meet the two circles in A, B and C, D respectively.

    Show that (*i*) $\triangle PAB \sim \triangle PCD$
    (*ii*) $AB \parallel CD$. [*C.B.S.E., 1984 (A.I.)*]

    Hint : Draw a tangent line to the circles at P.

12. In a $\triangle ABC$, a line segment PQ is drawn parallel to base BC intersecting AB and AC in P and Q respectively. Prove that the circumcircles of the $\triangle ABC$ and $\triangle APQ$ touch each other at A.

    Hint : At A, draw a tangent line to the circle ABC.

13. PP′ and QQ′ are two direct common tangents to two circles intersecting in points A and B. The common chord AB produced intersects PP′ in R and QQ′ in S. Prove that $RS^2 = PP'^2 + AB^2$.

14. AB is a line segment and M is its mid-point. Semi-circles are drawn with AM, MB and AB as diameters on the same side of the line AB. A circle C (O, *r*) is drawn so that it touches all the three semi-circles. Prove that $AB = 6r$.

REVIEW EXERCISE VII

(Section A)

1. Fill in the blanks to make each of the following statements true :

(a) The line through a point on a circle.........to the radius through the point, is the tangent line to the circle at that point. [*C.B.S.E., 1987 (A.I.)*]

(b) The lengths of the two tangents from an external point to a circle are.........
[*C.B.S.E., 1986 (A.I.); 1984 (A.I.) ; 1986 (Delhi)*]

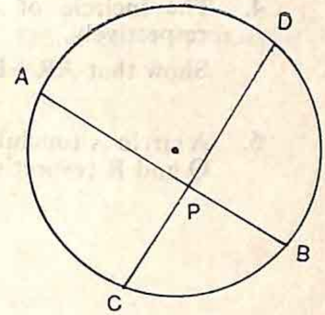
(c) A line which touches two given circles is called a.........to the circles.
[*C.B.S.E., 1986 (A.I.)*]

(d) If two circles touch each other, the point of contact lies on the line joining their.........

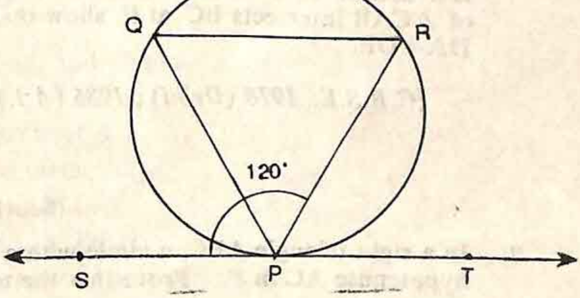
(e) Two circles C (O, r) and C (O′, s) will touch each other externally, if and only if, OO′ is equal to.........

(f) If two circles intersect in two points, they will have only one pair of.........common tangents.

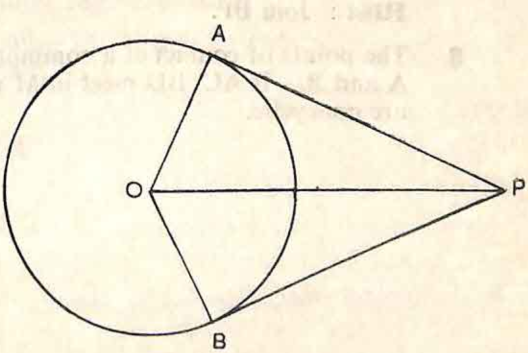
(g) In the figure, if AP=4 cm, BP=3 cm, and CP=2 cm, then PD=.........cm.
[*C.B.S.E., 1987 (Delhi)*]



(h) In the figure, SPT is a tangent to the circle and ∠SPR=120°, then ∠RQP=.........
[*C.B.S.E., 1986 (A.I.)*]

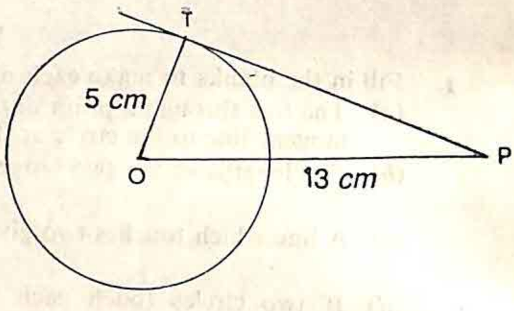


(i) PA and PB are tangents to a circle with centre O. If ∠OPA=30°, then ∠AOB is.........
[*C B.S.E., 1986 (Delhi)*]



(*j*) In the figure, if PT is a tangent to a circle whose centre is O, then length of PT is.........cm.

[*C.B.S.E., 1983 (Delhi)*]



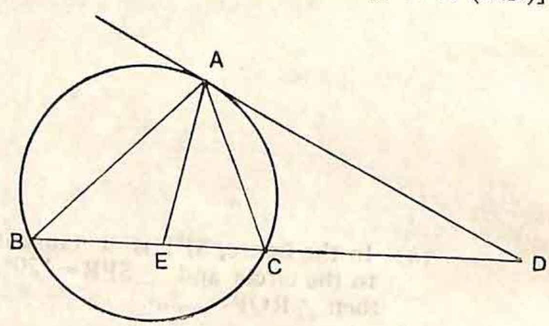
(Section B)

2. Prove that there is one and only one tangent at any point on the circle. [*C.B.S.E., 1986 (A.I.)*]
3. PQ and PR are equal chords of a circle. Prove that tangent at P to the circle is parallel to chord QR. [*C.B.S.E., 1986 (A.I.)*]
4. The incircle of a triangle ABC touches the sides BC, CA and AB at P, Q and R respectively.

   Show that $AR+BP+CQ=AQ+CP+BR=\frac{1}{2}$ (perimeter of $\triangle$ ABC)

   [*C.B.S.E., 1986 (Delhi); 1978 (Delhi)*]
5. A circle is touching side BC of a $\triangle$ABC at P and is touching AB and AC produced at Q and R respectively. Prove that AR is half the perimeter of $\triangle$ABC. [*C.B.S.E., 1987 (Delhi); 1986 (A.I.)*]
6. In the figure, DA is a tangent and DCB is a secant to a circle. If the bisector of $\angle$CAB intersects BC at E, show that DA=DE.

   [*C.B.S.E., 1978 (Delhi) ; 1986 (A.I.)*]



(Section C)

7. In a right triangle ABC, a circle with a side AB as diameter is drawn to intersect the hypotenuse AC in P. Prove that the tangent to the circle at P bisects the side BC.

   Hint : Join BP.
8. The points of contact of a common tangent to two circles which intersect at C and D are A and B. If AC, BD meet in M and AD, BC meet in L, prove that C, L, D and M are concyclic. [*C.B.S.E., 1986 (Delhi)*]

□ □

9

GEOMETRICAL CONSTRUCTIONS

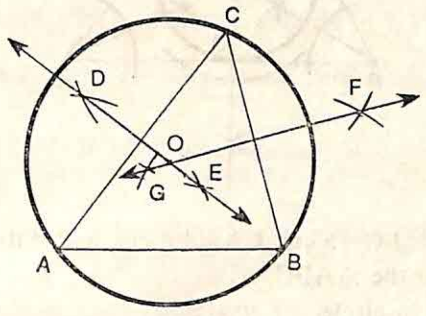
We have already studied about some geometrical constructions in Class IX. We shall now learn some more geometrical constructions involving circles.

9·1. CONSTRUCTION OF CIRCUMSCRIBED AND INSCRIBED CIRCLES OF TRIANGLES

CONSTRUCTION 24

To circumscribe a circle about a given triangle.

Construct a triangle with sides 35 mm, 32 mm and 40 mm. Draw the circumcircle of the triangle.



Given : Sides of a triangle are 35 mm, 32 mm and 40 mm.

Required : (*i*) To construct the triangle.

(*ii*) To draw the circumcircle of the triangle.

Const. : (*i*) (1) Construct $\triangle ABC$ with the given sides.

(*ii*) (1) Draw DE, the right-bisector of CA.

(2) Draw FG, the right-bisector of BC.

(3) Let these right-bisectors intersect each other at O.

(4) With O as centre and OA as radius draw a circle.

The circle passes through A, B and C and is the required circle.

Proof : Since O is on DE, the right-bisector of AC,

$$OA=OC \quad \ldots(1)$$

Since O is on FG, the right-bisector of BC,

$$OB=OC \quad \ldots(2)$$

From (1) and (2), we have $OA=OB=OC$

i.e., O is equidistant from A, B and C.

Hence a circle with centre O and radius equal to OA will pass through A, B and C.

The circle which passes through all the vertices of a triangle is called its circum-circle.

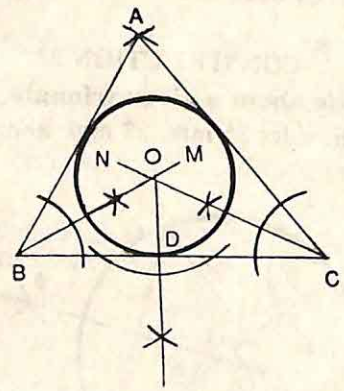
The radius of the circum-circle is called **circum-radius** and its centre is called **circum-centre.**

Note : The circum-centre of a triangle lies inside the triangle, on the mid-point of a hypotenuse, or outside the triangle, according as the triangle is an acute-angled, a right-angled or an obtuse-angled triangle.

CONSTRUCTION 25

To inscribe a circle in a given triangle.

Construct a $\triangle ABC$ such that $BC=5$ cm, $CA=4{\cdot}6$ cm, and $AB=3{\cdot}8$ cm. Inscribe a circle in the triangle.



Given : $\triangle$ ABC in which $BC=5$ cm, $CA=4{\cdot}6$ cm, and $AB=3{\cdot}8$ cm.

Required : (*i*) To construct the $\triangle ABC$.

(*ii*) To draw its in-circle.

Const. : (*i*) (1) Construct the $\triangle ABC$ with given sides.

(*ii*) (1) Draw BM, the bisector of $\angle B$.

(2) Draw CN, the bisector of $\angle C$.

(3) Let these bisectors intersect each other at O.

(4) From O, draw $OD \perp BC$.

(5) With centre O and radius OD, draw a circle.
The circle touches the sides of the $\triangle ABC$ and is the required circle.

The circle which is drawn inside the triangle so as to touch each of the sides is called its In-circle.

The radius of the in-circle is called the **In-radius** and its centre is called the **In-centre.**

EXERCISE 9 (a)

(Section A)

1. Perpendicular bisectors of the sides AB and AC of a triangle ABC meet in O.
   (*i*) What do you call the point O ?
   (*ii*) What is the relation between the distances OA, OB and OC ?
   (*iii*) Does the perpendicular bisector of BC pass through O ?
   (*iv*) What do you call the point where the perpendiculars drawn from the vertices A, B and C of the triangle ABC to their opposite sides meet ?

2. The bisectors of angles A and B of a scalene triangle ABC meet at O.

   (*i*) What is the point O called ?

   (*ii*) OR and OQ are drawn perpendicular to AB and CA respectively. What is the relation between OR and OQ ?

   (*iii*) What is the relation between $\angle$ACO and $\angle$BCO ?

3. Using ruler and compasses only inscribe a circle in the given triangle and measure its radius.

(Section B)

4. Construct a triangle with sides 26 mm, 28 mm and 30 mm. Circumscribe a circle about it and measure its radius.

5. Draw a circle passing through the vertices of an equilateral triangle whose side is 2·2 cm.

6. Construct a triangle ABC with BC=3 cm, $\angle$A=75°, $\angle$B=60°. Draw its circum-circle and measure its radius.

7. Draw a square with side 3·2 cm. Inscribe a circle in it.

8. Draw a triangle whose sides are 28 mm, 26 mm, and 30 mm. Inscribe a circle in it and measure its radius.

9. Construct a right-angled triangle having hypotenuse=2·6 cm, and a side=2·4 cm. Draw its in-circle and measure the in-radius.

10. Draw an equilateral triangle on a side of 4 cm and draw the inscribed and circumscribed circles. Find by measurement the radii of the two circles.

(Section C)

11. Draw a triangle whose sides are 31 mm, 42 mm and 53 mm. Draw its in-circle and circum-circle and measure the distance between the centres.

12. Using ruler and compasses only :

    (*a*) Construct a triangle ABC with the following data :

    Base AB=7 cm, BC=6·5 cm and $\angle$CAB=60°.

    (*b*) In the same figure, draw a circle which passes through the points A, B and C and mark its centre O.

    (*c*) Draw a perpendicular from O to AB which meets AB in D.

    (*d*) Prove that AD=BD.

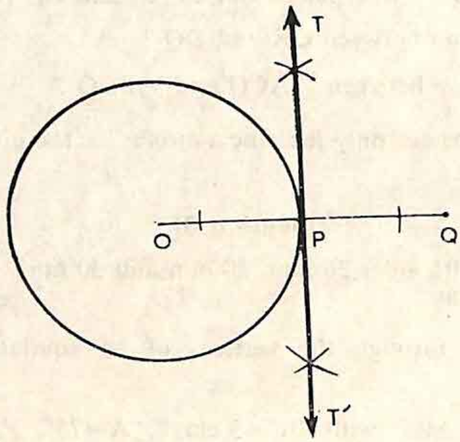
9·2. CONSTRUCTIONS OF TANGENTS

For the construction of tangents, the following properties which have been proved earlier, must be remembered.

1. Only one tangent can be drawn at a point on a circle.
2. A tangent is perpendicular to the radius through the point of contact.
3. Only two tangents can be drawn to a circle from a point outside the circle. They will be equal.

CONSTRUCTION 26

To draw a tangent to a given circle at a point on it.



Given : A circle with centre O.
A point P on it.

Required : To draw a tangent to the circle at P.

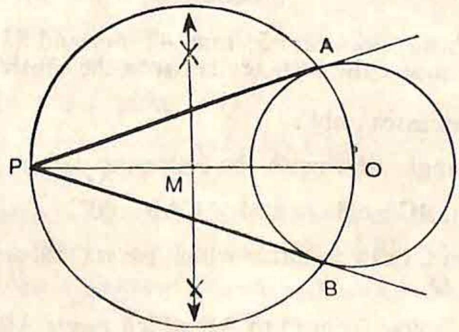
Const. : (1) Join OP and produce it to Q.

(2) At P, draw TPT′ $\perp$ OQ.

Then TPT′ is the required tangent line.

CONSTRUCTION 27

To draw tangents to a given circle from a point outside the circle.



Given : A circle with centre O.
A point P outside the circle.

Required : To draw tangents to the circle from P.

Const. : (1) Join OP.

(2) Bisect OP at M.

(3) With M as centre and MO as radius draw a circle cutting the given circle at A and B.

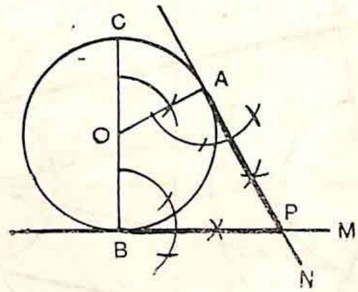
(4) Join PA and PB.

Then PA and PB are the two required tangents.

CONSTRUCTION 28

To draw tangents to a circle inclined at a given angle.

Draw a circle of radius 15 mm. Draw two tangents to it inclined at an angle of 60° to each other.



Steps of Construction :

(1) Draw any diameter BOC.

(2) Draw a radius OA making $\angle AOC = 60°$ (=the given $\angle$).

(3) At B, draw BM $\perp$ radius OB.

(4) At A, draw AN $\perp$ radius OA.

(5) Let these perpendiculars intersect each other at P.

Then PA and PB are the required tangents.

Note : In order to draw tangents to a given circle inclined at a given angle draw two radii making an angle equal to the supplement of the given angle at the centre. Then draw tangents ($\perp$s) at the ends of these radii.

EXERCISE 9 (b)

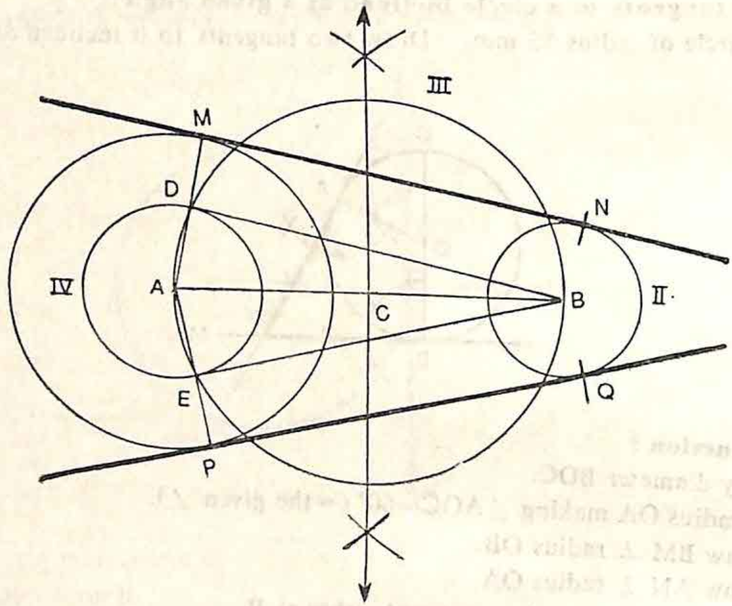
1. Draw a circle with the radius 1·9 cm and also draw a tangent to it at a point P on its circumference.
2. Draw a tangent to a circle of 4 cm as diameter from a point P outside the circle.
3. Using ruler and compasses only construct the tangents to the given circle from the point P. Measure the length of any one of them.
4. Draw a circle of radius 18 mm. Take a point P, 27 mm away from the centre. Draw tangents to the circle from this point. Calculate the length of the tangents.
5. Draw a circle of radius 2·4 cm. Draw two tangents to the circle such that the angle between them is equal to (*a*) 30°, (*b*) 45°.
6. From a point P on the circumference of a circle of radius 3 cm, draw a chord whose distance is 2 cm from the centre.
7. Draw a circle of radius 4 cm. Mark its centre as C and mark a point D such that CD=7 cm. Using ruler and compasses only, construct the two tangents from D to the circle. Measure the length of one of them.
8. Draw a circle of radius 3 cm. Take a point P, 5 cm away from its centre. From P draw tangents to the circle and measure the length of the tangent. [*C.B.S.E.*, *1986* (*Delhi*)]

9·3. CONSTRUCTION OF COMMON TANGENTS

CONSTRUCTION 29

To draw direct common tangents to two given circles.

Given two circles of radii 2·1 cm and 1 cm with their centres 5 cm apart. Draw direct common tangents to the two circles and measure the length of the tangents.



Given : Two circles I and II with centres A and B, and with radii 2·1 cm and 1 cm. Distance between the centres (AB)=5 cm.

Required : To draw the direct common tangents to the circles.

Const. : (1) Bisect AB at C.

(2) With C as centre and radius AC, draw a circle (III).

(3) With centre A and radius=(2·1−1) cm or 1·1 cm (=the difference of radii of given circles), draw another circle (IV) cutting the circle (III) at D and E.

(4) Join AD and produce it to meet the circle (I) at M.

(5) Join AE and produce it to meet the circle (I) at P.

(6) With centres M and P and radius BD or BE, draw arcs cutting the circle (II) at N and Q respectively.

(7) Join MN and PQ.

Then MN and PQ are the required tangents.

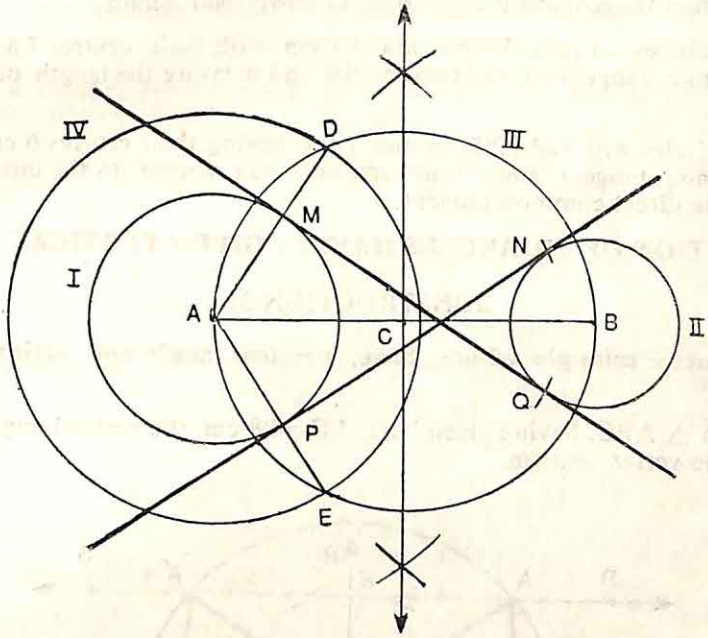
Measurement : MN=4·9 cm nearly and PQ=4·9 cm nearly.

Note : In step (6) the usual practice is to draw BN ∥ AM and BQ ∥ AP in the same sense. But in drawing parallels, accuracy is reduced. Hence, it is preferred to draw arcs as above.

CONSTRUCTION 30

To draw transverse common tangents to two given circles.

Given two circles of radii 1·6 cm and 1 cm with their centres 4·9 cm apart. Draw transverse common tangents to the circles and measure the length of the tangents.



Given : Two circles with the centres A and B and radii 1·6 cm and 1 cm respectively. Distance between the centres (AB)=4·9 cm.

Required : To draw the transverse common tangents to the circles.

Const. : (1) Bisect AB at C.

(2) With C as centre and radius AC, draw a circle (III).

(3) With A as centre and radius=(1·6+1) cm or 2·6 cm (=the sum of the radii of given circles) draw another circle (IV) cutting the circle (III) at D and E.

(4) Join AD cutting the circle (I) at M.

(5) Join AE cutting the circle (I) at P.

(6) With centres M and P and radius BD or BE, draw arcs cutting the circle (II) on the other side at Q and N respectively.

(7) Join MQ and PN.

Then MQ and PN are the required tangents.

Measurements : MQ=4·1 cm nearly, PN=4·1 cm nearly.

EXERCISE 9 (c)

1. Draw two circles with radii 2 cm and 3 cm with their centres 7 cm apart.

(*a*) Draw a direct common tangent and a transverse common tangent.

(*b*) Calculate the length of the direct common tangent.

2. Find the length of the direct common tangent to two circles of radii 3 cm and 5 cm with their centres 10 cm apart.

3. Draw two circles of radii 1·6 cm and 3 cm respectively, and with centres 5·6 cm apart. Draw the direct common tangents. Measure their lengths.

4. Draw two circles of radii 3·6 cm and 1·4 cm, their centres being 5·4 cm apart. Draw the transverse common tangents to these circles. Measure the length of these tangents.

5. Construct two equal circles with radius 1·8 cm each and centres 4·4 cm apart. Draw direct common tangents to them and write down their length.

6. Draw two circles of radii 4·7 cm and 3·1 cm with their centres 7·8 cm apart. Draw all the common tangents to the two circles and measure the length of a direct common tangent.

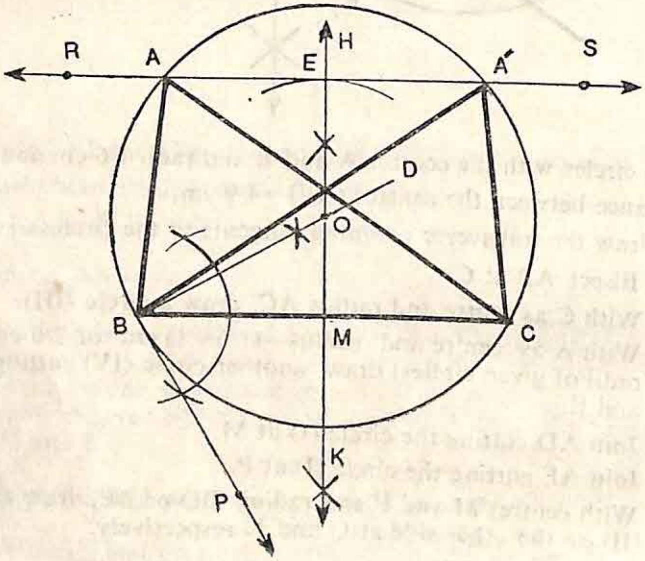
7. Draw two circles with radii 2·5 cm and 3 cm having their centres 6 cm apart. Draw a direct common tangent and a transverse common tangent to the circles. Measure the length of the direct common tangent.

9·4. CONSTRUCTION OF TRIANGLES HAVING GIVEN VERTICAL ANGLE

CONSTRUCTION 31

To construct a triangle whose base, vertical angle and altitude through the vertex are given.

Construct a $\triangle$ ABC, having given base BC=4·8 cm, the vertical angle A=60° and the altitude through the vertex =3 cm.



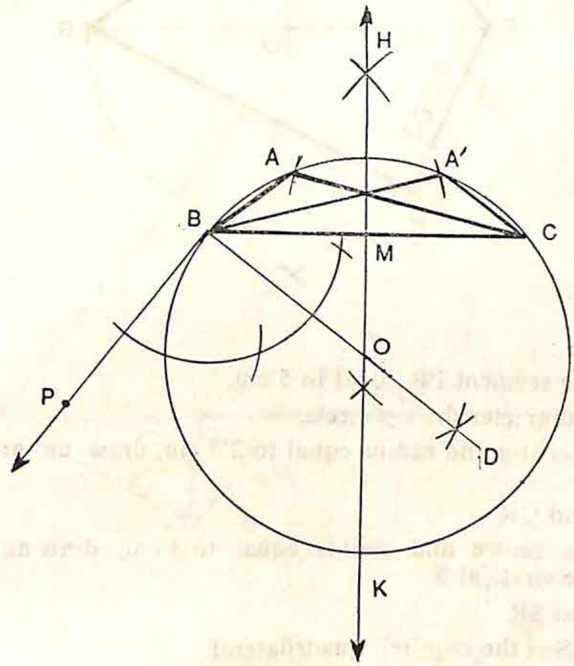
Steps of Construction :

(1) Draw a line segment BC equal to 4·8 cm.
(2) Make $\angle$PBC=60°.
(3) Draw line HK, the right bisector of BC, intersecting BC in M.
(4) Draw BD $\perp$ BP.
(5) Let BD and HK intersect at a point O.
(6) With O as centre and OB as radius, draw a circle.
[*Observe that the segment BAA′C contains an angle of 60°*]
(7) On the line HK mark segment ME=3 cm (length of the altitude).
(8) Through E, draw line RS $\perp$ line HK so that RS $\parallel$ BC.
(9) Let line RS intersect the circle at A and A′.
(10) Join AB, AC and A′B, A′C.
Then $\triangle$ ABC or $\triangle$ A′BC is the required triangle.

CONSTRUCTION 32

To construct a triangle whose base, vertical angle and median through the vertex are given.

Construct a triangle, having given the base = 4·5 cm, vertical angle = 120° and the median through the vertex = 1·5 cm.



Steps of Construction :

(1) Draw a line segment BC equal to 4·5 cm.

(2) Make $\angle PBC = 120°$.

(3) Draw line HK, the right-bisector of BC, intersecting BC in M.

(4) Draw $BD \perp BP$.

(5) Let BD and HK intersect at a point O.

(6) With O as centre and OB as radius draw a circle.

(7) With M, the mid-point of BC, as centre and radius equal to 1·5 cm draw arcs intersecting the circle at A and A′.

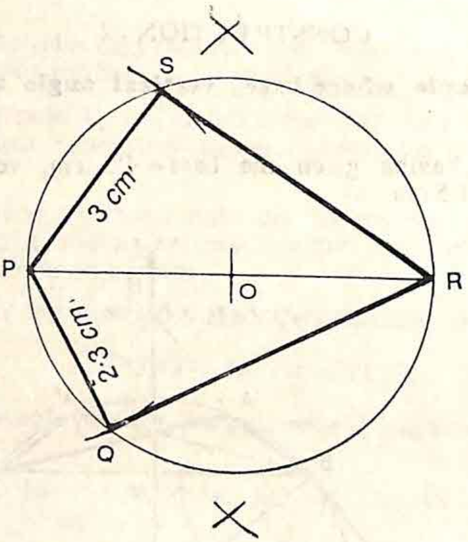
(8) Join AB, AC and A′B, A′C.

Then ΔABC or $\Delta A'BC$ is the required triangle.

CONSTRUCTION 33

To construct a cyclic quadrilateral with one vertex angle as a right angle.

Construct a cyclic quadrilateral PQRS in which PR = 5 cm, PQ = 2·3 cm, PS = 3 cm and $\angle S = 90°$.



Steps of Construction :

(1) Draw a line segment PR equal to 5 cm.

(2) On PR as diameter draw a circle.

(3) With P as centre and radius equal to 2·3 cm, draw an arc intersecting the circle at Q.

(4) Join PQ and QR.

(5) With P as centre and radius equal to 3 cm, draw an arc opposite to Q intersecting the circle at S.

(6) Join PS and SR.

Then PQRS is the required quadrilateral.

EXERCISE 9 (d)

1. Construct a ΔABC, having given base $BC=4{\cdot}5$ cm, the vertical angle $A=60°$ and one of the sides$=3{\cdot}3$ cm.
2. Construct a cyclic quadrilateral ABCD in which $AC=5$ cm, $AB=2$ cm, $CD=3$ cm and $\angle B=90°$.
3. Construct a triangle ABC in which $BC=5$ cm, $\angle A=50°$ and altitude through $A=4$ cm.
4. Construct a triangle ABC in which $BC=6$ cm, $\angle A=70°$, and median through $A=4$ cm.
5. Construct a ΔABC, having given base $BC=4$ cm, $\angle A=60°$, and altitude through the vertex$=2{\cdot}8$ cm.
6. Construct a triangle having base$=3{\cdot}5$ cm, vertical angle 40° and median through the vertex 2·5 cm.
7. Construct a cyclic quadrilateral ABCD in which $AC=6$ cm, $\angle ABC=55°$, $AB=2{\cdot}3$ cm and $AD=3$ cm.
8. Construct a cyclic quadrilateral ABCD with the following data :

   diagonal $AC=4$ cm, $\angle ABC=135°$, $AB=CD=3$ cm.

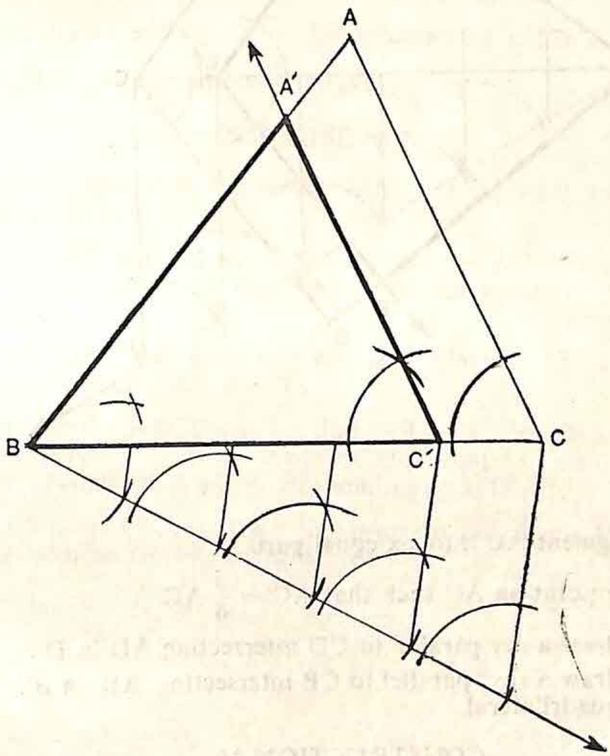
9·5. CONSTRUCTION OF SIMILAR FIGURE

We have learnt about similarity of two figures. We shall now construct some geometrical figures similar to given figures.

CONSTRUCTION 34

To construct a triangle similar to a given triangle as per the given scale factor.

Construct a triangle similar to a given triangle ABC with its sides equal to $\frac{4}{5}$th of the corresponding sides of ΔABC.



Steps of Construction :

(1) Divide the base BC of the $\triangle$ABC into five equal parts.

(2) Let C′ be a point on BC such that $BC' = \frac{4}{5}$ BC.

(3) Through C′, draw a ray parallel to CA intersecting BA at A′.
Then $\triangle$A′BC′ is the required triangle.

Proof. Line segments A′C′ and AC are parallel to each other.

$\therefore$ $\triangle$A′BC′ and $\triangle$ABC are equiangular.

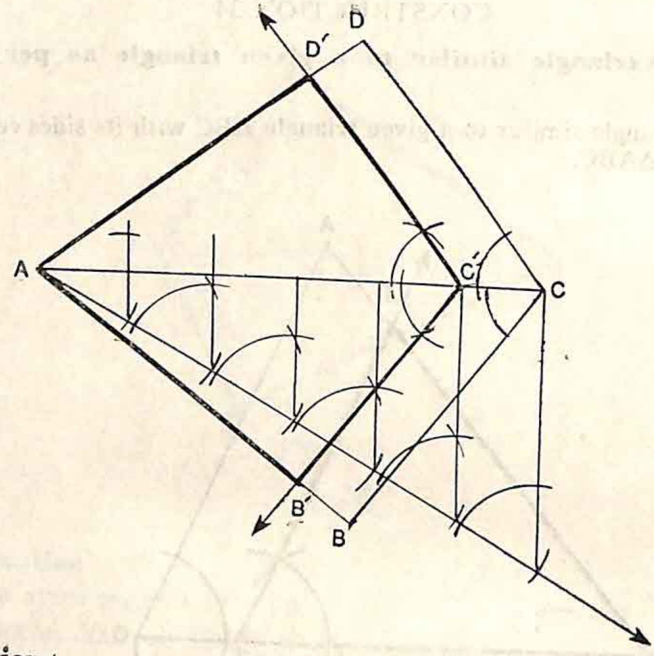
$$\therefore \quad \frac{A'B}{AB} = \frac{BC'}{BC} = \frac{C'A'}{CA} = \frac{4}{5}$$

i.e., $\triangle$A′BC′ is similar to $\triangle$ABC.

CONSTRUCTION 35

To construct a quadrilateral similar to a given quadrilateral as per the given scale factor.

Construct a quadrilateral similar to a given quadrilateral ABCD with its sides $\frac{5}{6}$th of the corresponding sides of ABCD.



Steps of Construction :

(1) Join AC.

(2) Divide line segment AC into six equal parts.

(3) Let C′ be the point on AC such that $AC' = \frac{5}{6} AC$.

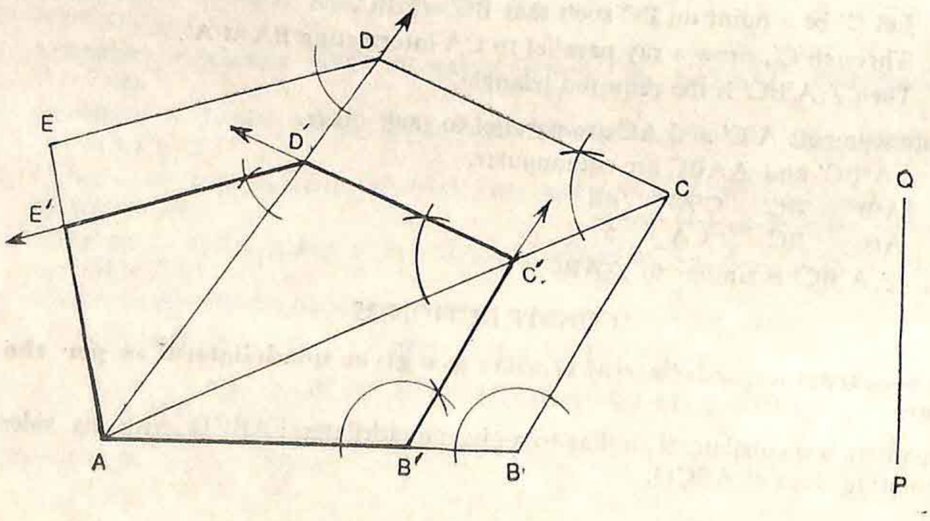
(4) Through C′, draw a ray parallel to CD intersecting AD in D′.

(5) Through C′, draw a ray parallel to CB intersecting AB in B′. Then AB′C′D′ is the required quadrilateral.

CONSTRUCTION 36

To draw a polygon similar to a given polygon on a side of given length.

Construct a pentagon similar to given pentagon ABCDE with side corresponding to AB and equal to PQ.



Steps of Construction :

(1) Join AC and AD.

(2) Mark a point B′ on AB such that AB′ = PQ.

(3) Through B′, draw a ray parallel to BC intersecting AC in C′.

(4) Through C′, draw a ray parallel to CD intersecting AD in D′.

(5) Through D′, draw a ray parallel to DE intersecting AE in E′.

Then AB′C′D′E′ is the required pentagon.

EXERCISE 9 (e)

1. Construct a triangle similar to a given $\triangle$ ABC with its sides equal to $\frac{3}{4}$th of the corresponding sides of $\triangle$ABC.
2. A triangle ABC is such that AB = 4 cm, BC = 5 cm and AC = 6 cm. Construct a triangle similar to the given $\triangle$ABC such that each of its side is $\frac{2}{3}$rd of the corresponding side of the given triangle.
3. Construct a quadrilateral similar to a given quadrilateral ABCD with its sides $\frac{4}{5}$th of the corresponding sides of quad. ABCD.
4. Draw a quadrilateral ABCD such that AB = 4·5 cm, BC = 4·8 cm, CD = 4 cm, DA = 4·3 cm and $\angle A = 75°$. Draw another quadrilateral similar to the given quadrilateral on a line segment of 3 cm, corresponding to side AB.
5. Construct any pentagon ABCDE. On a line segment PQ, 3· cm long, construct another pentagon PQRST similar to the pentagon ABCDE.

REVIEW EXERCISE VIII

(Section A)

1. For each of the following statements write 'True' or 'False' as appropriate :

   (*a*) A direct common tangent can always be drawn to any two given circles.

   (*b*) The centre of the circumcircle of a right-angled triangle is the mid-point of its hypotenuse.

   (*c*) The centre of incircle of a given triangle is always in the interior of the circle.

   (*d*) A figure similar to any rectilinear figure can be drawn by dividing the latter into triangles and then constructing similar triangles to these.

(Section B)

2. Construct a circle with centre O and radius as 2·5 cm. Draw a tangent at any point on the circle.

3. Use ruler and compasses to draw two tangents to a circle of radius 2·5 cm from a point P at a distance of 6 cm from its centre. Write also the steps of construction. [*C.B.S.E., 1987 (A.I.)*]

4. Construct a circle of radius 3 cm. Draw two tangents to this circle such that the angle between the tangents is 60°.

5. Using ruler and compasses, draw a tangent at a point P of a circle of radius 4 cm without using the centre. Also write the steps of construction. [*C.B.S.E., 1988 (Delhi)*]

6. Using ruler and compasses, draw a $\triangle ABC$ in which $BC=7$ cm, $CA=5$ cm and $AB=6$ cm and construct its circumcircle. Also write the steps of construction. [*C.B.S.E., 1987 (Delhi)*]

(Section C)

7. The centres of two circles of radii 2 and 3 cm respectively are 7 cm apart. Draw the pair of direct tangent lines to the two circles. Use ruler and compass only and write also the steps of construction. [*C.B S.E., 1988 (A.I.)*]

8. Construct a $\triangle ABC$ with the vertical angle $C=60°$, base $AB=5·4$ cm and altitude through $C=3·8$ cm.

9. Construct a $\triangle ABC$ having given that $BC=5$ cm, $\angle A=60°$ and the median bisecting the base equal to 3·7 cm. Use the ruler and compass only.

10. Construct a quadrilateral ABCD in which $AB=4·5$ cm, $BC=5·7$ cm, $CD=6$ cm, $DA=7·2$ cm and $BD=8$ cm. On a line segment 5 cm long, construct another quadrilateral similar to the given quadrilateral.

□□

10

STATISTICS

10·1. MEAN

A graph of data helps us to draw some conclusions about the data. The study of diagrams and graphs however depends much upon judgement and skill of drawing inferences. Sometimes it is helpful to inspect the data more closely for other worthwhile information *e.g.*, representative number. One valuable bit of information is provided by the central tendency of the data.

All of us are well aware of '*average*' which gives us the characteristic of a group and helps us in locating the central value. We speak of average rainfall, average temperature, batting average, etc., in daily life.

10·2. MEAN OF UNGROUPED DATA

Mean of *ungrouped data* is obtained by adding all the observations and dividing the sum by the total number of observations.

$$\text{Mean}=\frac{x_1+x_2+x_3+\cdots+x_n}{n}=\frac{\sum\limits_{i=1}^{n}x_i}{n},$$

where Σ (sigma) is used to denote the sum,

x_i is a single score, and n is the number of scores.

Mean is usually represented by M or $\bar{x}$.

Thus $$\bar{x}=\frac{\sum\limits_{i=1}^{n}x_i}{n}.$$

Example 1. Find the mean weight of 8 men, having their weights in kilograms as under :

65, 75, 70, 63, 62, 73, 64, 68.

Solution. Here $n=8$

$$\text{Mean}=\frac{\sum\limits_{i=1}^{n}x_i}{n}$$

$$\bar{x}=\frac{65+75+70+63+62+73+64+68}{8}$$

$$=\frac{540}{8}\qquad=67{\cdot}5\text{ kg}$$

This single representative number 67·5 kg. is called the '**Average**' or '**Arithmetic Mean**' or simply '**Mean**'.

This single number is capable of giving us the idea as to the weight of students in that class. So this is also called the *central tendency of the group*.

It should be noted that the *unit of measurement* in which the variable has been measured should be mentioned along with the value of the mean.

EXERCISE 10 (a)

(Section A)

1. The monthly income of six persons in a family is given below in rupees :
 460, 1200, 1420, 240, 1060, 300.
 Find the average monthly income in rupees.
2. The maximum temperature during seven days of the fourth week of May in New Delhi, were as :
 30°C, 29°C, 31°C, 36°C, 37°C, 39°C, 41°C.
 Calculate the mean maximum temperature for that week.
3. Find the mean weight of 8 men, having their weights in kilograms as under :
 70, 64, 67, 75, 72, 66, 65, 77.
4. Heights of ten persons in centimetres are given below :
 163, 156, 168, 158, 160, 164, 178, 167, 161, 170.
 Find their mean height.
5. The arithmetic mean of 3, 7, 5, x, 8, -3 is 4.
 Find x.
6. The mean of the numbers 6, y, 7, x, 14 is 8. Express y in terms of x.

(Section B)

7. The mean of 5 numbers is 27. If one number is excluded, their mean is 25. Find the excluded number. [*C.B.S.E., 1984 (Delhi)*]
8. The mean height of 20 students is 155 cm. It is discovered later on that while calculating the mean, the reading 149 cm was wrongly read as 189 cm. Find the correct mean. [*C.B.S.E., 1987 (A.I.)*]
9. Duration of sun-shine (in hours) in Delhi for first ten days of August, 1984 as reported by Meteorological Department are given below :
 5·1, 4·7, 6·1, 1·7, 5·4, 5·0, 11·7, 11·6, 11·5, 3·2.
 (*a*) Calculate $\bar{x}$.
 (*b*) Check that $\sum_{i=1}^{10} (x_1-\bar{x})=0$. [*C.B.S.E., 1987 (A.I.)*]

10·3. MEAN OF DISCRETE SERIES

Let us now learn how to find the mean of *grouped data—discrete series.*

If $x_1, x_2, x_3, \ldots\ldots, x_n$ are the values of variables and $f_1, f_2, f_3, \ldots\ldots, f_n$ are the corresponding frequencies, then

$$\text{Mean}=\frac{f_1x_1+f_2x_2+f_3x_3+\cdots+f_nx_n}{f_1+f_2+f_3+\cdots f_n}$$

Thus $$\bar{x}=\frac{\sum_{i=1}^{n} f_ix_i}{\sum_{i=1}^{n} f_i}$$

The following example will illustrate the method of calculation of the mean.

Example 2. Daily wages of 39 workers in a factory are given below :

Daily wages (in rupees)	Number of workers
7	4
8	7
10	11
11	8
14	6
15	3

Find their average daily wages.

Solution :

Daily wages (in rupees) x_i	*Number of workers* f_i	$f_i x_i$
7	4	28
8	7	56
10	11	110
11	8	88
14	6	84
15	3	45
Total	39	411

Here $\Sigma f_i x_i = 411, \quad \Sigma f_i = 39$

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \frac{411}{39}$$

$$= 10{\cdot}54$$

$\therefore$ average daily wages = Rs. 10·54.

Remember the following steps for computation of mean :

(1) *Multiply each value of the item* (x_i) *by its frequency* (f_i) *and write the product in the third column.*

(2) *Find the sum of these products* ($f_i x_i$).

(3) *Find the sum of the frequencies.*

(4) *Divide the sum of the products* ($\Sigma f_i x_i$) *by the sum of the frequencies* (Σf_i).

EXERCISE 10 (b)

(Section A)

1. Compute the arithmetic mean of the following frequency table :

Marks	Number of students
1	2
2	7
3	17
4	9
5	5

2. Following are marks (out of five) of 40 students :

Marks	0	1	2	3	4	5
Number of students	6	3	10	10	8	3

3. Daily wages of 39 workers in a factory are given below :

Daily wages (in rupees)	7	8	10	11	14	15
Number of workers	4	7	11	8	6	3

Find their average daily wages.

4. Find the mean age from the following data :

Age in years	Number of students
12	15
13	14
14	22
15	11
16	9
17	6
18	3

(Section B)

5. Compute the mean for marks obtained by 30 students in Mathematics in the Half-Yearly Examination.

The distribution of marks is given below :

Marks	Number of students
52	7
58	5
60	4
65	6
68	3
70	3
75	2

6. The heights in contimetres of 60 students are given below :

Height	156	157	158	159	160	161	162	163
Number of students	4	9	10	14	18	7	5	3

Find their average height to the nearest cm.

7. The following table gives the basic salaries of persons employed in a factory :

Salary in Rupees	Number of persons
110	5
130	7
150	10
170	15
190	13
210	16
230	14

Calculate the mean basic salary.

10.3. MEAN OF CONTINUOUS SERIES

For a continuous series, mean is calculated with the formula given below :

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$$

where x_i is the mid-value of a class-interval and f_i is the frequency of that class.

Remember the following steps for calculation of mean :

(1) *Find the mid value of each class-interval and write it in the third column.*

(2) *Multiply each mid value by the corresponding frequency.*

(3) *Write these products in the fourth column and find their sum.*

(4) *Divide this sum by the total of all the frequencies.*

Example 3. Find the arithmetic mean for the following distribution :

Weekly wages (in rupees)	Number of workers
12·50—17·50	2
17·50—22 50	22
22·50—27·50	19
27·50—32·50	14
32·50—37·50	3

Solution :

Weekly wages in rupees	*Number of workers* f_i	x_i	$f_i x_i$
12·50—17·50	2	15	30
17·50—22·50	22	20	440
22·50—27·50	19	25	475
27·50—32·50	14	30	420
32·50—37·50	3	35	105
Total	60	—	1470

Here $\Sigma f_i x_i = 1470$; $\Sigma f_i = 60$

$$\therefore \quad \bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \frac{1470}{60} = 24{\cdot}50$$

Hence average weekly wages = Rs. 24·50.

EXERCISE 10 (c)

(Section A)

1. Calculate the mean for the following table :

Age in years	Number of persons
0—10	4
10—20	8
20—30	12
30—40	32
40—50	14

2. Find the arithmetic mean from the frequency distribution given below :

Weekly wages (in rupees)	Number of workers
0—10	1
10—20	8
20—30	10
30—40	5
40—50	4
50—60	2

3. The frequency-distribution of ages of some persons in an office is given below :

Age in years	Number of persons
5—15	4
15—25	7
25—35	11
35—45	9
45—55	5
55—65	3

Find the mean of their ages.

(Section B)

4. Calculate the Arithmetic mean, correct to one decimal place, for the following frequency distribution of marks obtained in an Arithmetic Test :

Marks	0—10	10—20	20—30	30—40	40—50
Number of student	2	5	20	8	7

5. Find the mean for the following distribution :

Class-interval	0—8	8—16	16—24	24—32	32—40	40—48
Frequency	8	7	16	24	15	7

6. The ages of workers in a company are as follows :

Age (in years)	Number of workers
18—24	6
24—30	8
30—36	12
36—42	8
42—48	4
48—54	2

Calculate the average age of the group. [*C.B S.E., 1986 (A.I.)*]

7. In a study on certain disease, the following data were obtained :

Age at first detection (in years)	Number of patients
4—8	2
8—12	12
12—16	15
16—20	25
20—24	18
24—28	12
28—32	3
32—36	1

Find the average age at first detection. [*C.B.S.E., 1984 (A. I.)*]

10·4. SHORT-CUT METHOD FOR COMPUTING MEAN

When the frequencies and the values of the variable are quite large, a shorter method is used to save labour and time. We take an *arbitrary mean*. Then we find the deviations of the x-values from the arbitrary mean. The mean is calculated with the help of formulae given below :

For ungrouped data :

$$\bar{x}=A+\frac{\sum_{i=1}^{n} d_i}{n}$$

where A=the assumed mean,

$d_i=x_i-A$=deviation of the item from A

and n=number of items.

For discrete series :

$$\bar{x}=A+\frac{\sum_{i=1}^{n} f_i d_i}{\sum f_i}$$

where A=the assumed mean,

$d_i=x_i-A$=deviation of the item from A

f_i=frequency of the item

For continuous series :

$$\bar{x}=A+\frac{\sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i}$$

where A=the assumed mean,

$d_i=x_i-A$=deviation of the mid-value from A

and f_i=frequency of that class.

Example 4. Heights of 10 children in cm in a Nursery class are 56, 67, 61, 68, 65, 60, 58, 65, 63, 64. Find the mean height.

Solution : Let the assumed mean (A) be 60.

Heights in cm x_i	*Deviation* $d_i = x_i - A$
56	−4
67	7
61	1
68	8
65	5
60	0
58	−2
65	5
63	3
64	4
Total	27

Here $A=60$ $\quad \Sigma d_i=27$ $\quad n=10$

$$\bar{x}=A+\frac{\sum_{i=1}^{n} d_i}{n} \quad =60+\frac{27}{10}$$

$$=60+2{\cdot}7 \quad =62{\cdot}7$$

$\therefore$ Mean height$=62{\cdot}7$ cm

Example 5. Find the Arithmetic Mean for the following distribution, giving the ages of 100 students of class IX in a public school in New Delhl.

Age in years	Number of students
11	8
12	27
13	33
14	20
15	12

Solution : Let the assumed mean (A) be 13.

Age in years x_i	*Number of students* f_i	*Deviations* $d_i=x_i-\text{A}$	f_id_i
11	8	-2	-16
12	27	-1	-27
13	33	0	0
14	20	1	20
15	12	2	24
Total	100	—	1

Here $\quad \text{A}=13, \quad \Sigma f_id=1, \quad \Sigma f_i=100$

$$\bar{x}=\text{A}+\frac{\sum_{i=1}^{n} f_id_i}{\sum_{i=1}^{n} f_i}=13+\frac{1}{100}$$

$$=13+0{\cdot}1 \qquad =13{\cdot}01$$

$\therefore$ Mean age $=13{\cdot}01$ years

Example 6. Calculate the mean for the following distribution, giving the monthly incomes of 188 workers in a factory.

Income inrupees	*Number of workers*	*Income in rupees*	*Number of workers*
120—130	5	170—180	36
130—140	11	180—190	28
140—150	21	190—200	16
150—160	26	200—210	7
160—170	34	210—220	4

Solution : Let the assumed mean (A) be 175.

Income in rupees	*Number of workers* f_i	*Mid-value* x_i	*Deviation* $d_i = x_i - A$	$f_i d_i$
120—130	5	125	−50	−250
130—140	11	135	−40	−440
140—150	21	145	−30	−630
150—160	26	155	−20	−520
160—170	34	165	−10	−340
170—180	36	175	0	0
180—190	28	185	10	280
190—200	16	195	20	320
200—210	7	205	30	210
210—220	4	215	40	160
Total	188	—	—	−1210

Here $A = 175, \ \Sigma f_i d_i = -1210, \quad \Sigma f_i = 188$

$$x = A + \frac{\sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i} = 175 + \frac{-1210}{188}$$

$$= 175 - 6{\cdot}44 = 168{\cdot}56$$

Average monthly income = Rs 168·56.

EXERCISE 10 (d)

(Section A)

1. Find the mean of the following set of numbers 73, 71, 69, 67, 65, 63, 61, 59.
2. Following are the weights (in kg) of ten newly born babies in a hospital on a particular day :
   3·4, 3·6, 4·2, 4·5, 3·9, 4·1, 3·8, 4·5, 4·4, 3·6.
3. Heights of 10 children in cm in a Nursery class are 56, 67, 61, 68, 65, 60, 58, 65, 63, 64, Find the mean height.
4. Calculate the arithmetic mean of the 15 numbers given as 19, 24, 23, 22, 24, 23, 21, 22, 20, 23, 21, 20, 22, 23.

(Section B)

5. Find the arithmetic mean of the following data :

Milk in litres	Number of cows
2	25
3	15
4	10
5	5
6	5

6. Find the arithmetic mean for the following distribution, giving the ages of 100 students of class IX in a school :

Age in years	11	12	13	14	15
Number of students	8	27	33	20	12

7. In an Intelligence Test conducted in a school, the marks out of 50 were as under :

Marks	19	20	30	35	40
Number of students	5	2	18	3	5

Find the mean score.

8. Find the mean age from the following data :

Age in years	12	13	14	15	16	17	18
Number of students	10	15	16	14	8	5	2

(Section C)

9.

Marks	0—20	20--40	40—60	60—80	80—100
Number of students	6	12	22	7	3

For the distribution given above calculate the mean marks.

10. The data below gives the weekly earnings of 100 workers in a Flour Hill :

Weekly earnings (in rupees)	Number of workers
0—50	8
50—100	15
100—150	32
150—200	26
200—250	12
250—300	7

[*C.B.S.E., 1982 (Delhi)*]

11. Calculate the mean for the following distribution :

0—5	5	20—25	25
5—10	8	25—30	15
10—15	15	30—35	8
15—20	21	35—40	6

[*C.B.S.E., 1984 (Delhi)*]

12. Calculate mean for the following table :

Income-tax in rupees	*Number of persons paying*	*Income-tax in rupees*	*Number of persons paying*
20—30	7	60—70	21
30—40	9	70—80	8
40—50	14	80—90	5
50—60	17	90—100	3

13. The heights of 100 school children in the same age group observed and classified as below :

Height in cm	*Frequency*	*Height in cm*	*Frequency*
135—140	2	155—160	23
140—145	4	160—165	14
145—150	19	165—170	6
150—155	31	170—175	1

Find their average height.

14. Find the arithmetic mean for the following distribution giving wages af 80 workers :

Wages in rupees	*Number of persons*	*Wages in rupees*	*Number of persons*
70—80	3	110—120	20
80—90	5	120—130	16
90—100	10	130—140	7
100—110	15	140—150	4

15. Find the arithmetic mean from the following table :

Weekly wages in Rs	Number of workers
Below 10	1
Below 20	9
Below 30	19
Below 40	24
Below 50	28
Below 60	30

16. Find the mean of the following :

Marks	Number of Students
More than 60	0
More than 55	5
More than 50	11
More than 45	20
More than 40	40
More than 35	60
More than 30	70
More than 25	85
More than 20	90

10·5 MERITS AND DEMERITS OF MEAN

Merits :

(1) It is uniquely defined *i.e.* it has one and only one value.
(2) It is based on all observations.
(3) It is easy to compute.
(4) It is easily understood.

Demerits :

(1) It is very much affected by the extreme values.
(2) It cannot be computed if one of the observations of the data is missing.
(3) It cannot be computed uniquely if the classes at the end are open.
(4) It does not communicate all the information contained in the raw data.

Consider a series of observations. Then study the effect of

(*i*) adding 'k' to every item,

(*ii*) subtracting '*k*' from every item,

(*iii*) multiplying every item by '*k*',

(*iv*) dividing every item by '*k*',

on the arithmetic mean of the series.

What do you observe?

The arithmetic mean of any number of items is increased or decreased by the quantity k, if each of the items is increased or decreased by the same quantity k.

The arithmetic mean of any number of items is multiplied or divided by the quantity k, if each of the items is multiplied or divided by the same quantity k.

10·6. MEDIAN

If all the values of a variable are arranged in *ascending* or *descending* order of their magnitude, then the value of the middle term is called **median.**

Median of a distribution is the value of the variable which divides it into *two* equal parts.

For Ungrouped Data :

The median of ungrouped data should be computed in the following steps :

(*i*) *Arrange the items in ascending or descending order of magnitude.*

(*ii*) *Then value of the middle item is the required median.*

Let n be the number of items.

If n is odd, the value of $\left(\frac{n+1}{2}\right)$th item is taken as the median.

If n is even, we find the values of $\left(\frac{n}{2}\right)$th and $\left(\frac{n}{2}+1\right)$th items. The average of these two values gives the median.

Example 7. The heights in centimetres of nine girls are

70, 59, 67, 61, 63, 64, 61, 72, 71

Find the median height.

Solution. The height (in centimetres) arranged in increasing order are

59, 61, 61, 63, 64, 67, 70. 71, 72

Here the middle most value is 64 which is the median.

$\therefore$ median height is 64 cm.

Example 8. Find the median of the following numbers :

5, 7, 9, 12, 10, 8, 7, 15, 21, 19, 25, 11, 15, 18.

Solution. Arranging these numbers in ascending order,

5, 7, 7, 8, 9, 10, 11, 12, 15, 15, 18, 19, 21, 25.

Here $n=14$, which is an even number.

So there are two central items *i.e.*, $\left(\frac{14}{2}\right)$th and $\left(\frac{14}{2}+1\right)$th terms or 7th and 8th terms.

$$\therefore \quad \text{median} = \frac{\text{value of the 7th term} + \text{value of the 8th term}}{2}$$

$$= \frac{11+12}{2} = \frac{23}{2} = 11{\cdot}5.$$

EXERCISE 10 (e)

(Section A)

1. In a school examination it is decided that exactly half the pupils will pass. Name the measures of central tendency that is used.
2. A player makes the following scores in 8 matches :
   47, 41, 50, 39, 45, 48, 42, 48.
   Find this median score.
3. Find the median for the set of numbers :
   2, 2, 3, 5, 5, 5, 6, 8, 9.
4. Find the median of the following marks obtained by 16 students in a class-test marked out of 10 marks :
   0, 0, 2, 2, 3, 3, 3, 4, 5, 5, 5, 5, 6, 6, 7, 8.
5. The daily earnings of 10 workers in a factory are
   16, 8, 19, 7, 12, 6, 13, 14, 17, 16.
   Find the median earnings.

(Section B)

6. Find the median of the following observations :
   46, 64, 87, 41, 58, 77, 35, 90, 55, 92, 33.
   If 92 is replaced by 99 and 41 by 43 in the above data, find the new median.
   [*C.B.S E., 1987* (*A.I.*)]
7. {1, 2, 3, 6, 8} is a set of five positive integers whose mean is 4 and median is 3. Write down two other sets of five positive integers, each having the same mean and median as this set.

Like arithmetic mean, median too has some good properties :

(1) It is easy to compute.

(2) It is easily understood.

(3) It is rigidly defined.

(4) Unlike mean, it is not affected by the extreme values.

10·7. MORTALITY TABLES

Vital statistics deal with all the events of human life which have to do with an individual's entrance into or departure from life. So, vital statistics are *numerical records* of births, death, marriages, divorce separation, etc. In a broader sense vital statistics refer to all types of population statistics. For growth of population study of *mortality* factor is very important. Mortality rates help the health authorities in planning for sanitary improvements and improved medical facilities.

Study the following table showing death rate in India during 1941-1986 :

Period	*Death rate per 1000*
1941—1950	27·4
1951—1960	22·8
1961—1970	19·0
1971—1980	15·0
1981—1986	12·2

What inferences can you draw from the above table ?

During 1941-1950, death rate was 27·4 which means that out of 1000 persons nearly 28 died due to various factors. After independence death rate came down due to increase of

medical facilities and improvement in sanitary conditions. During 1971-1980, death rate came down to 15 per 1000. Note that death rate is a *fluctuating* measure. It can increase in a particular year due to natural calamity or epidemics.

10·8. INDEX NUMBER

The wages of factory workers are decided from time to time on the basis of the *cost of living* with the help of Index Numbers. Dearness allowance of Government employees is increased when *consumer price index* goes up. Index numbers are special types of weighted averages.

The index number is a widely used statistical device for comparing one group of related variables with another group of the same variables, for different periods of time, or places, between like categories. It is nothing but a representative number.

There are three types of index numbers which are commonly computed. They are

(1) the price index numbers,

(2) the quantity index numbers, and

(3) the cost of living index numbers.

You know that a barometer is used to measure atmosphere pressure or pressure of gases. Similarly, index numbers are used to measure the pressure of economic behaviour. Therefore, index numbers are called '*economic barometers*'.

The price index number is a numerical value that summarizes price levels.

Let us consider the following data :

Commodities	*Prices in 1985*	*Prices in 1988*
Wheat	Rs. 2·75 per kg	Rs. 3·50 per kg
Sugar	Rs. 5·25 per kg	Rs. 6·50 per kg
Tea	Rs. 25 per kg	Rs. 32 per kg
Butter	Rs. 48 per kg	Rs. 58 per kg
	Rs. 81	Rs. 90

We see that the total of the current prices for all commodities is Rs. 90 as against Rs. 81 in 1985 (*base year*).

$\therefore$ Prices in 1988 as compared with the prices in 1985 was $\frac{90}{81} \times 100$ *i.e.* 111·1 %

Thus, the **price index** of 1988 is 111 which means there is a net *increase* of 11% in the prices of commodities in the year 1988 as compared to 1985.

The whole sale price index numbers provide the information on general price levels in a country and the purchasing power of money.

The quantity index number is a measure of changing production or consumption.

Let us choose a middle-class working family in Delhi and find out monthly expenditure on certain items of consumption in 1985 and 1988.

Let us assume that there was no change in consumption pattern in that family.

Commodities	Quantity	1985		1988	
		Price per kg. in rupees	*Expenditure in rupees*	*Price per kg. in rupees*	*Expenditure in rupees*
Wheat	50 kg	2·75	137·50	3·50	175·00
Sugar	10 kg	5·50	55·00	6·50	65·00
Butter	3 kg	48·00	144·00	58·00	174·00
Tea	1 kg	25·00	25·00	32·00	32·00
Potatoes	30 kg	2·00	60·00	3·00	90·00
Total	—	—	421·50	—	536·00

We see that the middle-class working family had to spend Rs. 536·00 in 1988 as against Rs. 421·50 in 1985 to buy the same quantities of the same commodities.

Thus, the cost in 1988 as compared with cost in 1985 was $\frac{536}{421 \cdot 5} \times 100$ *i.e.* 127·2%. We say that *cost of living index* was 127 in 1988 with 1985 as *base year.*

The cost of living index number is a price index with special reference to a class or category of people in a society at different times or in different regions.

The cost of living index numbers provide guidelines to the government for deciding policies relating prices, wages, rent control, taxation, etc.

These numbers are used in wage negotiations, D.A. adjustments and grant of bonus to workers, etc.

EXERCISE 10 (f)

(Section A)

1. Fill in the blanks correctly :
   (*a*) In a broader sense vital statistics refer to all types of............statistics.
   (*b*) The population rates are calculated per............persons.
   (*c*) Index numbers are............of economic activity.
   (*d*) Usually an index number measures changes in a variable over a period of............
2. What do you understand by Index Numbers ?

(Section B)

3. Construct the index number for 1988 taking 1984 as the base year from the data given below :

Commodities	Prices in 1984 per quintal in rupees	Prices in 1988 per quintal in rupees.
A	70	100
B	50	80
C	100	150
D	230	270
E	190	200

4. Construct the cost of living index number for 1987 taking 1984 as base year, using the data about the monthly expenditure of a government employees family :

Items	Quantity	Prices per kg in 1984	Prices per kg in 1987
Wheat	20 kg	Rs. 2·50	Rs. 3
Rice	6 kg	Rs. 7	Rs. 6
Gram	5 kg	Rs. 1·60	Rs. 2·40
Pulses	8 kg	Rs. 5	Rs. 6
Ghee	2 kg	Rs. 25	Rs. 22
Sugar	10 kg	Rs. 3	Rs. 3
Huose Rent	—	Rs. 80	Rs. 100

REVIEW EXERCISE IX

(Section A)

1. Fill in the blanks to make each of the following a true statement :

(a) If the arithmetic mean of 6, 8. 5, 7, x and 4 is 7, then the value of x is............ [*C.B.S.E., 1987 (Delhi)*]

(b) The mean of the numbers 994, 996, 998, 1000, 1002 is............ [*C.B S.E , 1986 (A.I)*]

(c) If the mean of the marks of five students is 33 and the mean of the marks of four of them is 32·5, the marks of the fifth students are............ [*C.B S.E., 1986 (Delhi)*]

(d) The arithmetic mean of first ten natural numbers is............ [*C.B.S.E., 1985 (A.I.)*]

(e) The arithmetic mean of the first five positive odd integers is............ [*C.B.S.E., 1985 (A.I.)*]

(f) The mean of the first five natural numbers is............ [*C.B.S.E., 1981 (Delhi)*]

(g) The weight (in kg) of 5 men are 65, 62, 69, 66, 61. The median is............ [*C.B.S.E., 1987 (A.I.)*]

(Section B)

2. The mean of the five positive integers 3, 5, 8, 9 and x is 7. Find the value of x.

3. Find the mean of the following marks obtained by 16 students in a class-test marked out of 10 marks :

0, 0, 2, 2, 3, 3, 3, 4, 5, 5, 5, 5, 6, 6, 7, 8.

4. The mean height of 15 students is 154 cm. It is discovered later on that while calculating the mean the reading 175 cm was wrongly read as 145 cm. Find the correct mean height. [*C.B.S.E., 1987 (Delhi)*]

5. Duration of sunshine (in hours) in Amritsar for the first ten days of August, 1985 was

5·1, 4·7, 3·1, 1·6, 1·7, 5·4, 5·0, 11·7, 11·6 and 11·5.

Calculate mean ($\bar{x}$) and show that $\sum_{i=1}^{10} (x_i-\bar{x})=0$

[*C.B.S.E., 1986 (Delhi)*]

6. The average weight of a set of p articles is q grams and the total weight of another set of r articles is s kg. Calculate the average weight of $(p+r)$ articles in kilograms.

7. Compute the mean of the following data :

Height (in cm)	Frequency	Height (in cm)	Frequency
219	2	204	7
216	4	201	5
213	6	198	4
210	10	195	1
207	11		

[*C.B.S.E., 1986 (Delhi)*]

(Section C)

8. In a study on diabetic patients, the following data are obtained :

Age at detection (in years)	Number of cases
10—19	1
20—29	0
30—39	1
40—49	10
50—59	17
60—69	38
70—79	9
80—89	3

Find the mean age at detection.

9. Compute the mean of the following frequency table by
(*i*) a direct-method and (*ii*) a short-cut method.

Class	Frequency
5—10	10
10—15	6
15—20	4
20—25	12
25—30	8
30—35	4
35—40	2
40—45	1
45—50	3

10. Calculate the cost of living index number for 1971 on the basis of 1960 from the following data :

Item	*Year 1960*		*Year 1971*
	Price/cost per unit	*Quantity (Weight) in units*	*Price/cost per unit*
Cereals	2·50	42·8	4·89
Other food articles	1·87	8·4	3·95
Clothing	3·68	7·5	5·24
Fuel and light	7·51	10·1	13·08
Rent	3·57	2·6	8·29
Education	5·29	3·9	7·96
Recreation	14·01	4·3	18·53
Miscellaneous	2·83	5·8	5·77

11. Calculate the mean of the following frequency distribution :

Income (in rupees)	Number of Workers
200−300	5
300−400	36
400−500	24
500−600	16
600−700	9
700−800	6
800−900	4

[*C.B.S.E., 1981* (*Delhi*)]

12. The data below gives the earning of 350 workers in a cotton mill. Find the average monthly earning of the group.

Monthly earnings (in rupees)	Number of Works
160−180	40
180−200	54
200−220	60
220−240	72
240−260	45
260−280	32
280−300	28
300−320	15
320−340	4

[*C.B.S.E., 1985* (*A.I.*)]

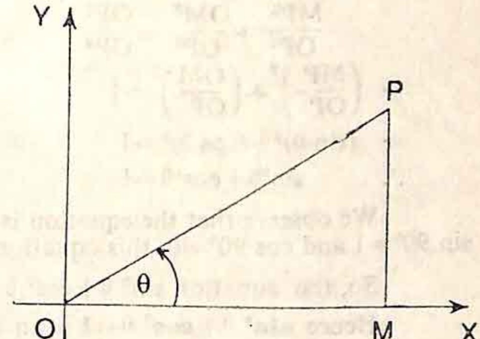
11

TRIGONOMETRY

11·1. REVIEW

In class IX, you learnt about trigonometric ratios defined with the help of a right triangle. You will now learn more about concepts and results in trigonometry.

Let us review them.



$$\sin\theta = \frac{\text{perpendicular}}{\text{hypotenuse}} = \frac{MP}{OP},$$

$$\cos\theta = \frac{\text{base}}{\text{hypotenuse}} = \frac{OM}{OP},$$

$$\tan\theta = \frac{\text{perpendicular}}{\text{base}} = \frac{MP}{OM},$$

$$\text{cosec}\,\theta = \frac{\text{hypotenuse}}{\text{perpendicular}} = \frac{OP}{MP}, \quad \sec\theta = \frac{\text{hypotenuse}}{\text{base}} = \frac{OP}{OM},$$

$$\cot\theta = \frac{\text{base}}{\text{perpendicular}} = \frac{OM}{MP}.$$

Angle	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined
cosec	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined
cot	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

11·2. TRIGONOMETRIC IDENTITIES

An equation in one or more variables is said to be an identity, if the equation is satisfied for all values of the variables involved.

In an identity, left hand side is equal to right hand side for all replacements of the variable(s) for which both sides are defined.

We shall now establish some basic trigonometric identities and use them to obtain some more simple identities.

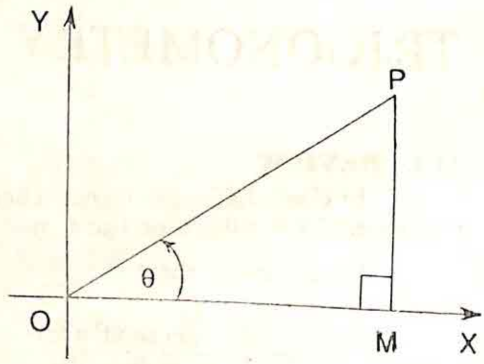
Let the revolving line OP, starting from OX trace out $\angle XOP = \theta$, in the positive direction.

From P, draw $PM \perp OX$.

Now we get a right-triangle POM.

So, by Pythagoras theorem, we have

$$MP^2 + OM^2 = OP^2$$



(1) Dividing both sides by OP^2, we get

$$\frac{MP^2}{OP^2} + \frac{OM^2}{OP^2} = \frac{OP^2}{OP^2}$$

$$\Rightarrow \left(\frac{MP}{OP}\right)^2 + \left(\frac{OM}{OP}\right)^2 = 1$$

$$\Rightarrow (\sin\theta)^2 + (\cos\theta)^2 = 1$$

$$\therefore \quad \sin^2\theta + \cos^2\theta = 1$$

We observe that the equation is defined for $0° < \theta < 90°$. Since $\sin 0° = 0$, $\cos 0° = 1$, $\sin 90° = 1$ and $\cos 90° = 0$, this equation is satisfied, if we replace θ by $0°$ or $90°$.

So, the equation $\sin^2\theta + \cos^2\theta = 1$ is satisfied for all values of the variable θ.

Hence $\mathbf{\sin^2\theta + \cos^2\theta = 1}$ is an identity.

(2) Dividing both sides by OM^2, we get

$$\frac{MP^2}{OM^2} + \frac{OM^2}{OM^2} = \frac{OP^2}{OM^2}$$

$$\Rightarrow \left(\frac{MP}{OM}\right)^2 + 1 = \left(\frac{OP}{OM}\right)^2$$

$$\Rightarrow (\tan\theta)^2 + 1 = (\sec\theta)^2$$

$$\therefore \quad \tan^2\theta + 1 = \sec^2\theta$$

i.e., $\mathbf{1 + \tan^2\theta = \sec^2\theta}$

(3) Dividing both sides by MP^2, we get

$$\frac{MP^2}{MP^2} + \frac{OM^2}{MP^2} = \frac{OP^2}{MP^2}$$

$$\Rightarrow 1 + \left(\frac{OM}{MP}\right)^2 = \left(\frac{OP}{MP}\right)^2$$

$$\Rightarrow 1 + (\cot\theta)^2 = (\text{cosec}\,\theta)^2$$

$$\therefore \quad \mathbf{1 + \cot^2\theta = \text{cosec}^2\theta}.$$

Thus, we have derived the following three fundamental identities :

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \text{cosec}^2\theta$$

We can derive some more identities from these identities by simple applications of elementary algebraic operations like addition, subtraction, multiplication and factorization. Some examples are given below :

(*i*) $\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$ $\qquad$ $\cos^2\theta = 1 - \sin^2\theta$

(*ii*) $1 + \tan^2\theta = \sec^2\theta$

$\tan^2\theta = \sec^2\theta - 1$ $\qquad$ $\sec^2\theta - \tan^2\theta = 1$

(*iii*) $1+\cot^2\theta=\operatorname{cosec}^2\theta$

$\cot^2\theta=\operatorname{cosec}^2\theta-1$ $\qquad\qquad$ $\operatorname{cosec}^2\theta-\cot^2\theta=1$

Example 1. Prove the following identity

$$\cos\theta\operatorname{cosec}\theta\tan\theta=1$$

Solution. L.H.S. $=\cos\theta\operatorname{cosec}\theta\tan\theta$, $\qquad$ for $0°<\theta<90°$

$$=\cos\theta\cdot\frac{1}{\sin\theta}\cdot\frac{\sin\theta}{\cos\theta}$$

$=1$

$=$ R.H.S.

Example 2. Prove that

$\cos^4\theta-\sin^4\theta+1=2\cos^2\theta$, for $0°\leqslant\theta<90°$.

Solution. L.H.S. $=\cos^4\theta-\sin^4\theta+1$, $\qquad$ for $0°\leqslant\theta<90°$

$=(\cos^2\theta)^2-(\sin^2\theta)^2+1$

$=(\cos^2\theta+\sin^2\theta)(\cos^2\theta-\sin^2\theta)+1$

$=(1)(\cos^2\theta-\sin^2\theta)+1$

$=\cos^2\theta-\sin^2\theta+1$

$=\cos^2\theta-(1-\cos^2\theta)+1$

$=\cos^2\theta-1+\cos^2\theta+1$

$=2\cos^2\theta$

$=$ R.H.S.

Example 3. Show that $\sec^4 A-\sec^2 A=\tan^2 A+\tan^4 A$, for $0°\leqslant A<90°$.

Solution. L.H.S. $=\sec^4 A-\sec^2 A$, $\qquad$ for $0°\leqslant A<90°$

$=\sec^2 A(\sec^2 A-1)$

$=(1+\tan^2 A)(1+\tan^2 A-1)$

$=(1+\tan^2 A)\tan^2 A$

$=\tan^2 A+\tan^4 A$

$=$ R.H.S.

EXERCISE 11 (a)

Prove the following identities :

(Section A)

1. $\cot\theta\sec\theta=\operatorname{cosec}\theta$
2. $\tan^2\theta\operatorname{cosec}\theta\cos^2\theta=\sin\theta$
3. $(\sin\theta+\cos\theta)^2=1+2\sin\theta\cos\theta$
4. $\cos^4\theta-\sin^4\theta=\cos^2\theta-\sin^2\theta$
5. $\tan^2\theta\cos^2\theta+\sin^2\theta\cot^2\theta=1$
6. $\dfrac{1-\tan\theta}{1+\tan\theta}=\dfrac{\cot\theta-1}{\cot\theta+1}$

(Section B)

7. $\cot\theta+\tan\theta=\operatorname{cosec}\theta\sec\theta$
8. $\sqrt{\operatorname{cosec}^2\theta-1}=\cos\theta\operatorname{cosec}\theta$
9. $\tan^4\theta+\tan^2\theta=\sec^4\theta-\sec^2\theta$
10. $\operatorname{cosec}^4\theta-\operatorname{cosec}^2\theta=\cot^4\theta+\cot^2\theta$

11. $\dfrac{1+\tan^2\theta}{1+\cot^2\theta}=\dfrac{\sin^2\theta}{\cos^2\theta}$

12. $\dfrac{1}{\sec A+\tan A}=\sec A-\tan A$

Hint : $\dfrac{1}{\sec A+\tan A}=\dfrac{1}{\sec A+\tan A}\times\dfrac{\sec A-\tan A}{\sec A-\tan A}$

(Section C)

13. $\dfrac{\sec\theta+\tan\theta}{\sec\theta-\tan\theta}=1+2\tan\theta\sec\theta+2\tan^2\theta$

14. $\cos^4 A+\sin^4 A-2\sin^2 A\cos^2 A=(1-2\sin^2 A)^2$ *[C.B.S.E., 1982 (A.I.)]*

15. $\sec^2\theta\,\mathrm{cosec}^2\theta=\cot^2\theta+\dfrac{1}{\cot^2\theta}+2$

Example 4. Prove that $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}=\sec\theta+\tan\theta$ *[C.B.S.E., 1982 (A.I.)]*

Solution. L.H.S. $=\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}$

$$=\sqrt{\frac{1+\sin\theta}{1-\sin\theta}\times\frac{1+\sin\theta}{1+\sin\theta}}$$

[*Multiply the numerator and denominator by the conjugate of the denominator.*

$$=\sqrt{\frac{(1+\sin\theta)^2}{1-\sin^2\theta}}$$

$$=\sqrt{\frac{(1+\sin\theta)^2}{\cos^2\theta}}$$

$$=\frac{1+\sin\theta}{\cos\theta}$$

$$=\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}$$

$$=\sec\theta+\tan\theta$$

= R.H.S.

Example 5. Prove that

$$\frac{\cos\theta}{1-\tan\theta}+\frac{\sin\theta}{1-\cot\theta}=\sin\theta+\cos\theta$$

Solution. L.H.S. $=\dfrac{\cos\theta}{1-\tan\theta}+\dfrac{\sin\theta}{1-\cot\theta}$

$$=\frac{\cos\theta}{1-\dfrac{\sin\theta}{\cos\theta}}+\frac{\sin\theta}{1-\dfrac{\cos\theta}{\sin\theta}}$$

$$=\frac{\cos\theta}{\dfrac{\cos\theta-\sin\theta}{\cos\theta}}+\frac{\sin\theta}{\dfrac{\sin\theta-\cos\theta}{\sin\theta}}$$

$$=\frac{\cos^2\theta}{\cos\theta-\sin\theta}+\frac{\sin^2\theta}{\sin\theta-\cos\theta}$$

$$=\frac{\cos^2\theta}{\cos\theta-\sin\theta}-\frac{\sin^2\theta}{\cos\theta-\sin\theta}$$

$$=\frac{\cos^2\theta-\sin^2\theta}{\cos\theta-\sin\theta}$$

$$=\frac{(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)}{\cos\theta-\sin\theta}$$
$$=\cos\theta+\sin\theta$$
$$=\sin\theta+\cos\theta$$
$$=\text{R.H.S.}$$

EXERCISE 11 (b)

Prove the following identities :

(Section A)

1. $\sqrt{\dfrac{1-\cos A}{1+\cos A}}=\dfrac{\sin A}{1+\cos A}$ [*C.B.S.E., 1983 (A.I.)*]
2. $\sqrt{\dfrac{1+\cos A}{1-\cos A}}=\operatorname{cosec} A+\cot A$ [*C.B.S.E., 1987 (A.I.)*]
3. $\sqrt{\dfrac{1-\sin A}{1+\sin A}}=\sec A-\tan A$
4. $\dfrac{1}{1+\operatorname{cosec}^2\theta}+\dfrac{1}{1+\sin^2\theta}=1$
5. $(\sin\theta+\cos\theta)(\cot\theta+\tan\theta)=\sec\theta+\operatorname{cosec}\theta$
6. $(\tan\theta\operatorname{cosec}\theta)^2-(\sin\theta\sec\theta)^2=1$

(Section B)

7. $\dfrac{2\sin\theta\cos\theta-\cos\theta}{1-\sin\theta+\sin^2\theta-\cos^2\theta}=\cot\theta$
8. $(1-\sin\theta+\cos\theta)^2=2(1-\sin\theta)(1+\cos\theta)$
9. $(\sin\theta+\operatorname{cosec}\theta)^2+(\cos\theta+\sec\theta)^2=\tan^2\theta+\cot^2\theta+7$

(Section C)

10. $\dfrac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta+1}+\dfrac{\operatorname{cosec}\theta}{\operatorname{cosec}\theta-1}=2\sec^2\theta$
11. $\dfrac{1}{\operatorname{cosec} A-\cot A}-\dfrac{1}{\sin A}=\dfrac{1}{\sin A}-\dfrac{1}{\operatorname{cosec} A+\cot A}$
12. $\dfrac{\tan\theta}{\sec\theta+1}+\dfrac{\tan\theta}{\sec\theta-1}=2\operatorname{cosec}\theta$
13. $\dfrac{\tan A}{1-\cot A}+\dfrac{\cot A}{1-\tan A}=\sec A\operatorname{cosec} A+1$

Example 6 : Examine whether the equation

$$\cot\theta-\tan\theta=\frac{2\cos^2\theta+1}{\sin\theta\cos\theta}$$

is an identity.

Solution. The variable θ in this equation can take values $0°\leqslant\theta<90°$. Let us substitute $\theta=45°$ in both sides.

$$\text{L.H.S.}=\cot\theta-\tan\theta=\cot 45°-\tan 45°$$
$$=1-1$$
$$=0$$

$$\text{R.H.S.}=\frac{2\cos^2\theta+1}{\sin\theta\cos\theta}=\frac{2\cos^2 45°+1}{\sin 45°\cos 45°}$$

$$=\frac{2\left(\frac{1}{\sqrt{2}}\right)^2+1}{\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}}=\frac{2\times\frac{1}{2}+1}{\frac{1}{2}}$$

$$=\frac{2}{\frac{1}{2}} \qquad =4$$

$\therefore$ L.H.S. $\neq$ R.H.S., when $\theta=45^\circ$.

This proves that the given equation is not an identity.

Example 7. Solve the equation $\cot\theta+\tan\theta=2\operatorname{cosec}\theta$

Solution. The variable θ in the equation can take such values as $0^\circ\leqslant\theta<90^\circ$.

$$\cot\theta+\tan\theta=2\operatorname{cosec}\theta$$

or $$\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}=\frac{2}{\sin\theta}$$

or $$\cos^2\theta+\sin^2\theta=2\cos\theta$$

or $1=2\cos\theta$ Then $\cos\theta=\frac{1}{2}$

The given equation is satisfied for $\theta=60^\circ$.

Hence, the solution of the given equation is $\theta=60^\circ$.

EXERCISE 11 (c)

(Section A)

Determine whether the following equations are identities :

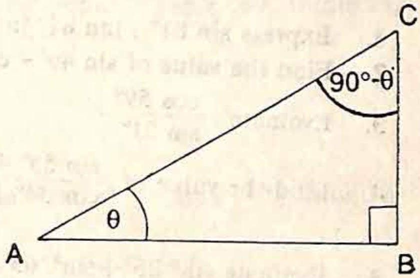
1. $\sin^2\theta+\sin\theta=1$
2. $\cot^2\theta+\cos\theta=\sin^2\theta$
3. $\dfrac{1-\tan^2\theta}{\cot^2\theta+1}=2\tan^2\theta$
4. $\dfrac{\tan\phi+\sin\phi}{\tan\phi-\sin\phi}=\dfrac{\sec\phi+1}{\sec\phi-1}$
5. $\tan^4\phi+\tan^6\phi=\tan^3\phi\sec^2\phi$

(Section B)

Solve the following equations :

6. $\dfrac{\cos\theta}{1-\sin\theta}+\dfrac{\cos\theta}{1+\sin\theta}=4$
7. $\tan^2\theta+\cot^2\theta=2$
8. $2\sin^2\theta-5\sin\theta+2=0$
9. $\tan^2\theta-\sec\theta-1=0$
10. $\dfrac{\cos^2\theta-3\cos\theta+2}{\sin^2\theta}=1$
11. $\dfrac{\cos^2\theta}{\cot^2\theta-\cos^2\theta}=3$
12. $\dfrac{\cos\theta}{\operatorname{cosec}\theta+1}+\dfrac{\cos\theta}{\operatorname{cosec}\theta-1}=2.$

11·3. TRIGONOMETRIC RATIOS OF COMPLEMENTARY ANGLES



Let ABC be a right triangle in which $\angle ABC = 90°$.
Let $\angle BAC = \theta$. Then $\angle ACB = 90° - \theta$

Here BC is the opposite side and AB is the adjacent side with respect to angle θ.

Then $\cos\theta = \dfrac{AB}{AC}$, for $0° \leqslant \theta < 90°$

AB is the opposite side and BC is the adjacent side with respect to angle $90° - \theta$.

Then $\sin(90° - \theta) = \dfrac{AB}{AC}$, for $0° \leqslant \theta < 90°$

So, $\mathbf{\sin(90° - \theta) = \cos\theta}$, for $0° \leqslant \theta \leqslant 90°$.

Observe that $\sin 0° = 0 = \cos 90°$ and $\sin 90° = 1 = \cos 0°$

Again, $\sin\theta = \dfrac{BC}{AC}$, for $0° \leqslant \theta < 90°$

and $\cos(90° - \theta) = \dfrac{BC}{AC}$, for $0° \leqslant \theta < 90°$

So, $\mathbf{\cos(90° - \theta) = \sin\theta}$, for $0° \leqslant \theta < 90°$.

Observe that $\cos 0° = 1 = \sin 90°$ and $\cos 90° = 0 = \sin 0°$

Thus, *cosine of an angle is the sine of the complementary angle and sine of an angle is the cosine of the complementary angle.*

In the same way, we can pair tangent with cotangent and secant with cosecant.

For $0° \leqslant \theta \leqslant 90°$, $\tan(90° - \theta) = \dfrac{\sin(90° - \theta)}{\cos(90° - \theta)} = \dfrac{\cos\theta}{\sin\theta} = \cot\theta$

$\therefore$ $\mathbf{\tan(90° - \theta) = \cot\theta}$

Similarly, $\cot(90° - \theta) = \dfrac{\cos(90° - \theta)}{\sin(90° - \theta)} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$

$\therefore$ $\mathbf{\cot(90° - \theta) = \tan\theta}$

For $0° \leqslant \theta \leqslant 90°$, $\sec(90° - \theta) = \dfrac{1}{\cos(90° - \theta)} = \dfrac{1}{\sin\theta} = \operatorname{cosec}\theta$

$\therefore$ $\mathbf{\sec(90° - \theta) = \operatorname{cosec}\theta}$

Similarly, $\operatorname{cosec}(90° - \theta) = \dfrac{1}{\sin(90° - \theta)} = \dfrac{1}{\cos\theta} = \sec\theta$

$\therefore$ $\mathbf{\operatorname{cosec}(90° - \theta) = \sec\theta}$

Example 8. Show $\sin 60° = \cos 30°$, $\sec 30° = \operatorname{cosec} 60°$ and $\tan 45° = \cot 45°$.

Solution. $\sin 60° = \sin(90° - 30°) = \cos 30°$

$\sec 30° = \sec(90° - 60°) = \operatorname{cosec} 60°$

$\tan 45° = \tan(90° - 45°) = \cot 45°$

Example 9. Find the value of

$\sin 20° \sin 70° - \cos 20° \cos 70°$

Solution. $\sin 20° \sin 70° - \cos 20° \cos 70°$

$= \sin 20° \sin(90° - 20°) - \cos 20° \cos(90° - 20°)$

$= \sin 20° \cos 20° - \cos 20° \sin 20°$

$= \sin 20° \cos 20° - \sin 20° \cos 20°$

$= 0$

EXERCISE 11 (d)

(Section A)

1. Express $\sin 81^\circ + \tan 81^\circ$ in terms of angles between 0° and 45°.
2. Find the value of $\sin 40^\circ - \cos 50^\circ$.
3. Evaluate $\dfrac{\cos 59^\circ}{\sin 31^\circ}$.
4. Find the value of $\dfrac{\sin 55^\circ\ 44'}{\cos 34^\circ\ 16'}$.

(Section B)

5. Evaluate $\sin^2 25^\circ + \sin^2 65^\circ$.
6. Prove that $\sin\theta \cos(90^\circ-\theta) + \cos\theta \sin(90^\circ-\theta) = 1$
7. Prove that $\cos 12^\circ \sin 78^\circ + \sin 12^\circ \cos 78^\circ = 1$
8. Prove that $\tan 9^\circ \tan 27^\circ \tan 45^\circ \tan 63^\circ \tan 81^\circ = 1$

(Section C)

9. Find x from the equation
$$x \sin(90^\circ-\theta) \cot(90^\circ-\theta) = \cos(90^\circ-\theta)$$
10. Prove that $\sin(90^\circ-A) \sin A \cot A - 1 = -\sin^2 A$ [*C.B.S.E*, *1978 (A.I.)*]
11. Prove that $\sin(90^\circ-\theta)\cos(90^\circ-\theta) = \dfrac{\tan\theta}{1+\tan^2\theta}$ [*C.B.S.E.*, *1980 (Delhi)*]
12. Compare the areas of the right triangles ABC and DEF in which $\angle A = 30^\circ$, $\angle B = 90^\circ$, AC = 4 cm, $\angle D = 60^\circ$, $\angle E = 90^\circ$ and DE = 4 cm.

11·4. TRIGONOMETRIC TABLES

You already know the values of trigonometric ratios of certain angles *e.g.* 0°, 30°, 45°, 60° and 90°. But you need trigonometric ratios of different angles in different practical situations. Values of different trigonometric ratios have been computed correct to certain places of decimals. Tables of trigonometric ratios of such angles have been printed on pages to.

These tables have been given into two forms. Most of the values in the tables are *approximate*. For all practical purposes, these approximate values are taken as actually correct values.

A degree is subdivided into 60 equal parts. Each part is called a *minute*.

So, 1 degree = 60 minute or $1^\circ = 60'$

Table I gives the values, correct upto three or four places of decimals, of all the six trigonometric ratios for angles from 0° to 90° at intervals of *10 minutes*. First and last columns have measures of angles. Between them are six columns headed by sin, cos, tan, cot, sec and cosec and containing respective values.

Example 10. Find the value of $\sin 38^\circ$.

Solution : We look in the first column for 38° in the Table I. Then we see further in the row containing 38° till we find 0·6157 in the vertical column headed by sin.

$$\therefore \quad \sin 38^\circ = 0{\cdot}6157$$

Example 11. From Table I, find the value of $\tan 50^\circ\ 20'$.

Solution. We look in the first column for $50^\circ\ 20'$. We now move in the row containing $50^\circ\ 20'$. Then we move down in the column headed by tan At the intersection of these, we find the number 1·206.

$$\therefore \quad \tan 50^\circ\ 20' = 1{\cdot}206.$$

Table II gives the values of all the trigonometric ratios, at intervals of $0{\cdot}1^\circ$ *i.e.*, $6'$ correct upto three or four places of decimals. The pattern of the table is similar to that of Table I in other respects.

Example 12. From Table II, find the value of see $(42{\cdot}6)^{\circ}$.

Solution. We look in the first column for $42{\cdot}6^{\circ}$ in the Table II. Then we read the number in the row containing $42{\cdot}6^{\circ}$ and in the column headed by secant. There we locate the number $1{\cdot}3585$.

$$\therefore \qquad \sec (42{\cdot}6)^{\circ}=1{\cdot}3585$$

Example 13. Find the value of $\cot (67{\cdot}4)^{\circ}$.

Solution. We locate the row which contains $(67{\cdot}4)^{\circ}$ in the first column. The number in this row and in the column headed by cot is $0{\cdot}4163$.

$$\therefore \qquad \cot (67{\cdot}4)^{\circ}=0{\cdot}4163.$$

EXERCISE 11 (e)

(Section A)

Using trigonometric tables, find the values of the following :

1. $\sin 35^{\circ}$ 2. $\cos 16^{\circ}$ 3. $\tan 37^{\circ}$
4. $\sin 55^{\circ}$ 5. $\cot 40^{\circ}$ 6. $\sec 63^{\circ}$
7. $\tan 65^{\circ}\, 20'$ 8. $\cos 31^{\circ}\, 40'$ 9. $\operatorname{cosec} 24^{\circ}\, 50'$
10. $\tan (16{\cdot}4)^{\circ}$ 11. $\sec (51{\cdot}4)^{\circ}$ 12. $\sin (25{\cdot}7)^{\circ}$

(Section B)

Use trigonometric tables to find ;

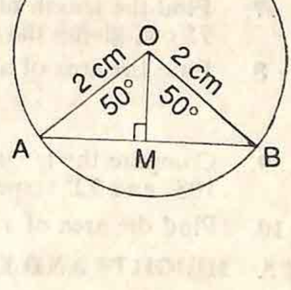
13. $\sin 64^{\circ}\, 42'+\cos 42^{\circ}\, 20'$ 14. $\tan 36^{\circ}\, 40'+\cot 63^{\circ}\, 20'$
15. $\tan 47^{\circ}\, 30'+\operatorname{cosec} 68^{\circ}\, 20'$.

Example 14. Find the length of the chord of a circle with radius 2 cm, subtending an angle of 100° at the centre of the circle. Give the answer correct to the first place of decimal.

Solution. Let O be the centre of the circle and AB be its given chord.

Then $OA=OB=2$ cm and $\angle AOB=100^{\circ}$

Recall that a perpendicular from the centre of a circle to a given chord bisects the chord.



Draw $OM \perp AB$. Then M is the mid-point of AB.

$\triangle AOM \cong \triangle BOM$ (R.H.S.—Congruency Theorem)

$\therefore \quad \angle AOM=\angle BOM=\frac{1}{2}\times 100^{\circ}$ *i.e.* 50°

In the right-angle AOM,

$$\sin AOM=\frac{AM}{OA} \Rightarrow \sin 50^{\circ}=\frac{AM}{2}$$

$$\therefore \qquad AM=2\times \sin 50^{\circ}=2\times 0{\cdot}7660=1{\cdot}5320 \text{ cm}$$

So, the length of the chord $AB=2AM$

$$=2\times 1{\cdot}5320 \text{ cm}$$

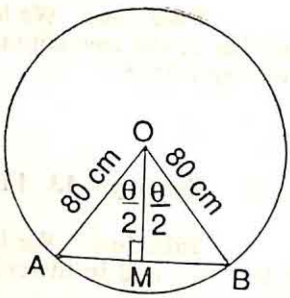
$$=3{\cdot}0640 \text{ cm} \qquad \textit{i.e., } 3{\cdot}1 \text{ cm}$$

Example 15. Find the length of a side of a regular polygon of 25 sides inscribed in a circle of radius 80 cm. Give the answer correct to the second decimal place.

Solution. If inscribed regular polygon has n sides, then each side subtends an angle θ at the centre,

where $\theta=\dfrac{360^\circ}{n}$

Here $\theta=\left(\dfrac{360}{25}\right)^\circ=(14{\cdot}4)^\circ$



Let AB be one of the sides of the inscribed regular polygon.
Join OA and OB. Draw OM $\perp$ AB.

Now $OA=OB=80$ cm,
$\angle AOB=\theta=(14{\cdot}4)^\circ$

Also OM bisects the chord AB.

Then $AM=MB$

And $\triangle AOM\cong\triangle BOM$ (R.H.S.—Congruency Theorem)

$\therefore$ $\angle AOM=\angle BOM$
$=\dfrac{\theta}{2}=(7{\cdot}2)^\circ$

In the right triangle AOM,

$\sin AOM=\dfrac{AM}{OA}$ $\Rightarrow$ $\sin(7{\cdot}2)^\circ=\dfrac{AM}{80}$

$\therefore$ $AM=80\times\sin(7{\cdot}2)^\circ=80\times0{\cdot}1253=10{\cdot}024$ cm

So, $AB=2\,AM=2\times10{\cdot}024$ cm $=20{\cdot}048$ cm

Thus, the length of the side $=20{\cdot}05$ cm, correct to the second decimal place.

EXERCISE 11 (f)

(Section A)

Find the length of the chord of a unit circle, subtending at the centre an angle of

1. 120° 2. 144° 3. 108°
4. Find the length of the chord of a circle with radius 2 cm, subtending an angle of 45° at the centre of the circle.
5. Find the length of the chord of a circle with radius 3 cm, subtending an angle of $(14{\cdot}6)^\circ$ at the centre of the circle.

(Section B)

6. Find the length of a side of a regular polygon inscribed in a circle of radius one metre, if it has 100 sides. Give the answer in cm correct to the first decimal place.
7. Find the length of a side of a regular polygon of 24 sides inscribed in a circle of radius 75 cm, giving the answer correct to the first decimal place.
8. Find the area of an isoceles triangle with base 10 cm and vertical angle 57°.

(Section C)

9. Compare the lengths of chords of circles with radii 3 cm and 4 cm subtending angles 108° and 72° respectively at the corresponding centres.
10. Find the area of a right triangle with hypotenuse 6 cm and one of the acute angles 77°.

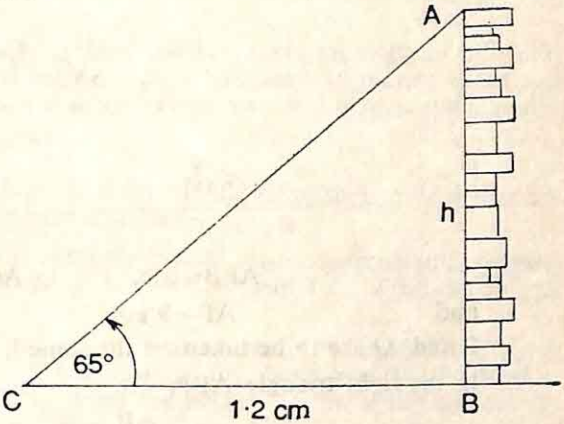
11·5. HEIGHTS AND DISTANCES

In the previous class you learnt how to solve simple problems on heights and distances using trigonometric ratios of special angles 30°, 45° and 60°. You will now solve similar problems involving different angles using *trigonometric tables* and *logarithmic tables*.

Example 16. A ladder is placed against a wall such that it just reaches the top of the wall. The foot of the ladder is 1·2 metres away from the wall and the ladder is inclined at an angle of 65° with the ground. Find the height of the wall correct upto two places of decimals.

Solution. Let AB represent the vertical height of the wall.

Let A and C be the positions of the top and bottom of the ladder, when AC represent the length of the ladder.



Let the height of the wall be h metres.

Now AB$=h$ metres, BC$=1·2$ metres,

$$\angle ACB=65° \text{ and } \angle ABC=90°$$

In the right triangle ABC,

$$\tan ACB=\frac{AB}{BC} \quad \Rightarrow \tan 65°=\frac{h}{1·2}$$

$$\therefore \quad h=1·2\times\tan 65°$$
$$=1·2\times 2·145$$
$$=2·574 \text{ metres}$$

Thus, the height of the wall

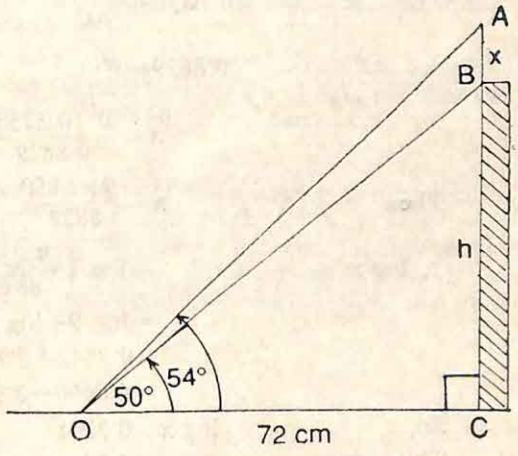
$=2·574$ metres *i.e.*, 2·57 metres

Example 17. An observer standing 72 metres away from a building notices that the angles of elevation of the top and the bottom of a flagstaff on the building are respectively 54° and 50°. Find the height of the flagstaff. [*C.B.S.E.*, *1978* (*A.I.*)]

Solution. Let AB and BC represent the vertical heights of the flagstaff and the building respectively where A is the top and B is the bottom of the flagstaff.

Let O be the point of observation such that $\angle AOC=54°$, $\angle BOC=50°$ and

$$OC=72 \text{ m}$$



Let the height of the building be h metres. Then BC$=h$ metres

In the right triangle BOC,

$$\tan BOC=\frac{BC}{OC} \quad \Rightarrow \quad \tan 50°=\frac{h}{72}$$

$$\therefore \quad h=72\times\tan 50°=72\times 1·192$$

Let the height of the flagstaff be x metres.

Then AB$=x$ metres

In the right triangle AOC,

$$\tan AOC=\frac{AC}{OC} \quad \Rightarrow \tan 54°=\frac{h+x}{72}$$

$$\therefore \quad h+x=72\times\tan 54°$$
$$=72\times 1·376$$

Then $x=72\times 1·376-h=72\times 1·376-72\times 1·192$
$$=72(1·376-1·192)$$
$$=72\times 0·184 \qquad =13·248 \text{ metres}$$

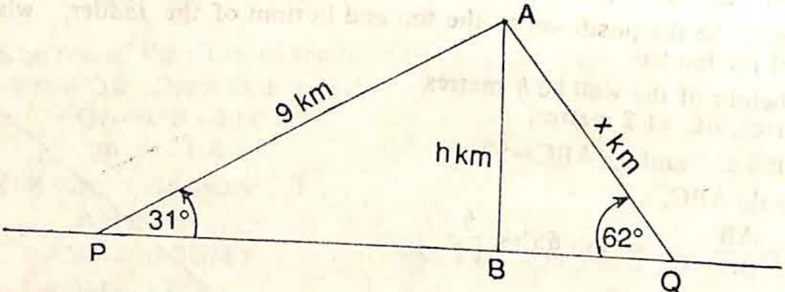
$\therefore$ the height of the flagstaff$=13·25$ metres

Example 18. The angles of elevation of the top of a hill at the city centres of two towns on either side of the hill are observed to be 31° and 62°. If the distance uphill from

the first city centre is 9 km, find in kilomètres, the distance uphill from the other city centre, correct upto two places of decimals.

Solution : Let AB represent the vertical height of the hill.

Let P and Q be the city centres on either side of AB such that



$\angle APB = 31°$, $\angle AQB = 62°$

and $AP = 9$ km

P and Q are to be taken on the same level ground. Let h km be the height of the hill.

In the right-triangle ABP,

$$\sin APB = \frac{AB}{AP} \Rightarrow \sin 31° = \frac{h}{9}$$

$$\therefore \quad h = 9 \sin 31° = 9 \times 0{\cdot}5150$$

Let the required distance uphill from the centre Q be x km.

In the right triangle ABQ,

$$\sin AQB = \frac{AB}{AQ} \Rightarrow \sin 62° = \frac{h}{x}$$

$$\Rightarrow \quad 0{\cdot}8829 = \frac{h}{x} \Rightarrow x = \frac{h}{0{\cdot}8829}$$

$$\Rightarrow \quad x = \frac{9 \times 0{\cdot}5150}{0{\cdot}8829}$$

Then $$x = \frac{9 \times 5150}{8829}$$

$$\therefore \log x = \log\left(\frac{9 \times 5150}{8829}\right)$$

$$= \log 9 + \log 5150 - \log 8829$$

$$= 0{\cdot}9542 + 3{\cdot}7118 - 3{\cdot}9459$$

$$= 4{\cdot}6660 - 3{\cdot}9459$$

So, $\log x = 0{\cdot}7201$

This, gives $x = 5{\cdot}249$

Thus the required distance $= 5{\cdot}249$ km

$= 5{\cdot}25$ km

correct upto two places of decimals.

EXERCISE 11 (g)

(Section A)

1. The angle of elevation of the top of a tower at a distance 40 metres from its foot on a horizontal plane is found to be 57°. Find the height of the tower.

2. A glider is flying at an altitude of 792 m The angle of depression of the control tower of an airport from the glider is 18° 40′. Find the horizontal distance between the glider and control tower.

3. The string of a kite is 90 metres long and it makes an angle of 48° with the horizontal. Find the distance of the shadow of the kite from the holder.

4. Find the distance of the observer from the top of a cliff which is 132 metres high, given that angle of the elevation is 41°18′.

5. The upper part of a tree, broken by the wind in two parts, makes an angle of 31° with the ground. The top of the tree touches the ground at a distance of 9 metres from the foot of the tree. Find the height of the tree in metres correct upto 2 places of decimals.

(Section B)

6. Two masts are 15 metres and 10 metres high, and the line joining their tops makes an angle of 33°41′ with the horizontal. Find their distance apart.

7. A person walking along a straight road observes that at two consecutive kilometre stones the angles of elevation of a hill in front of him are 30° and 75°. Find the height of the hill.

8. A vertical tower stands on a horizontal plane and is surmounted by a flagstaff of height 7 m. At a point on the plane, the angle of elevation of the bottom of the flagstaff is 36° and that of the top of the flagstaff is 45°. Find the height of the tower correct up to two places of decimal.

9. The angles of depression of two ships on either side of a light-house as observed from the top of the light-house are 52° and 41° 40′ respectively. If the height of light-house is 150 metres, find the distance between the two ships.

10. A fire at a building B is reported on telephone to two fire stations F_1 and F_2, 10 km apart from each other on a straight road. F_1 observes that the fire is at an angle of 50° to the road and F_2 observes that it is at an angle of 45° from it. Which station should send his team and how much will it have to travel ?

11. A person, standing on the bank of a river observes that the angle subtended by a tree on the opposite bank is 64°, when he retires 40 metres from the bank, he finds the angle to be 35°. Find the height of the tree and the breadth of the river.

(Section C)

12. The tallest tower in a city is 100 m high and a multistoreyed hotel at city centre is 20 m high. The angle of elevation of the top of the tower at top of the hotel is 3° 36′. A building h metres high, is situated on the road connecting the tower with the city centre at a distance of 1 km from the tower. Find the value of h, if the top of the hotel, the top of the building and the top of the tower are in a straight line. Also find the distance of the tower from the city centre.

REVIEW EXERCISE X

(Section A)

1. Fill in the blanks to make the following statements true :

(*a*) Value of $\sin 55° - \cos 35°$ is..... [*C.B.S.E., 1983 (A.I.)*]

(*b*) $\frac{\cos 53°}{\sin 37°} = \ldots\ldots$ [*C.B.S.E., 1983 (Delhi)*]

(*c*) The value of $\sin\theta \sin(90°-\theta) - \cos\theta \sin(90°-\theta)$ is........ [*C.B.S.E., 1982 (Delhi)*]

(*d*) The value of $\frac{\cos 20° 35'}{\sin 69° 25'}$ is........

(*e*) The value of $\cos\theta \cos(90°-\theta) - \sin\theta \sin(90°-\theta)$ is.......... [*C.B.S E., 1981 (A.I.)*]

(*f*) The value of $\sin^2 20° + \sin^2 70°$ is...... [*C.B.S.E., 1979 (A.I.)*]

2. Evaluate (*a*) $\frac{\sin 40°}{\cos 50°}$ (*b*) $\cos^2 17 - \sin^2 73°$.

(Section B)

Prove the following identities :

3. $1 - 2\sin\theta\cos\theta = (\sin\theta - \cos\theta)^2$. [*C.B.S E., 1983 (A.I.)*]

4. $\frac{1}{1+\sin A} + \frac{1}{1-\sin A} = \frac{2}{\cos^2 A}$ [*C.B.S.E , 1984 (A.I.)*]

5. $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \frac{\cos\theta}{1+\sin\theta}$ [*C.B.S.E., 1980 (Delhi)*]

6. $\frac{\sin(90°-A)\cos(90°-A)}{\tan A} = 1 - \sin^2 A$

7. $\frac{1}{\operatorname{cosec}\theta + \cos\theta} = \operatorname{cosec}\theta - \cot\theta$ [*C.B.S.E., 1978 (A.I.)*]

8. An electric pole is 10 m high. A steel wire tied to the top of the pole and affined at a point on the ground to keep the pole vertical, makes an angle of 49° with the horizontal line through the foot of the pole. Find the length of the steel wire.

9. Find the length of a side of a regular polygon inscribed in a circle of radius one metre, if it has 24 sides. Give the answer in cm correct to the first decimal place.

(Section C)

10. Prove that $\frac{\cos A}{1-\tan A} + \frac{\sin^2 A}{\sin A - \cos A} = \sin A + \cos A$ [*C.B.S E., 1983 (Delhi)*]

11. Prove that $\frac{\tan^2\theta}{\tan^2\theta - 1} + \frac{\cos^2\theta}{\sin^2\theta - \cos^2\theta} = \frac{1}{\sin^2\theta - \cos^2\theta}$

12. A surveyer noted that the angle of elevation of a marker on top of a hill was 15° 20′. He walked 40 metres towards the foot of the hill along level ground and found the angle of elevation of the marker as 30° 30′. How far from surveyor's first position was the marker ?

13. A vertical wall and a 40 metres high tower are in the same horizontal plane. From the top of the tower, the angles of depression of the top and bottom of the wall are 24° 30′ and 48° 20′ respectively. Find the height of the wall.

14. Prove that $\sin^6\theta+\cos^6\theta=1-3\sin^2\theta\cos^2\theta$

Hint. $(a+b)^3=a^3+b^3+3ab(a+b)$

$\Rightarrow \quad a^3+b^3=(a+b)^3-3ab(a+b)$

Now $\sin^6\theta+\cos^6\theta=(\sin^2\theta+\cos^2\theta)^3-3\sin^2\theta\cos^2\theta(\sin^2\theta+\cos^2\theta)$

15. Prove that $2(\sin^6\theta+\cos^6\theta)-3(\sin^4\theta+\cos^4\theta)+1=0$

16. Prove that $\dfrac{\tan A+\sec A-1}{\tan A-\sec A+1}=\dfrac{1+\sin A}{\cos A}$

Hint. $$\text{L.H.S.}=\frac{(\tan A+\sec A)-(\sec^2 A-\tan^2 A)}{\tan A-\sec A+1}$$

$$=\frac{(\sec A+\tan A)(1-\sec A+\tan A)}{\tan A-\sec A+1}.$$

□□

12
COMPUTING–II

FLOWCHARTING

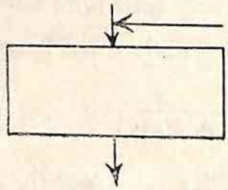
INTRODUCTION

You have already become familiar with the words 'computing' and 'computer'. As you know computer is an electronic machine that can store and process both numeric and non-numeric information and its important characteristics are (*i*) It carries out instructions most obediently : (*ii*) It does not get tired or bored by monotonous, lengthy and repetitive type of work and (*iii*) It works very fast.

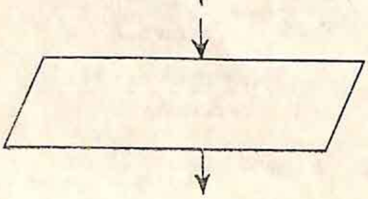
One thing should be clear that computer cannot do any work on its own. Whenever we want the computer to solve a problem, we must provide it, in a suitable form, a method of solving that problem. In other words a computer has to be programmed for solving a particular problem. Before we proceed any further let us do quick revision of what we did last year.

Algorithm. A series of step-by-step instructions that leads to the solution of a problem is called an algorithm. Algorithm is thus a design or a plan of obtaining a solution to a problem. Computer science, computing science, Informatics, computing etc. are different names used to represent more or less the same body of knowledge. Algorithm is so much the central concept of this body of knowledge that some experts even use the name 'algorithmics for this subject.

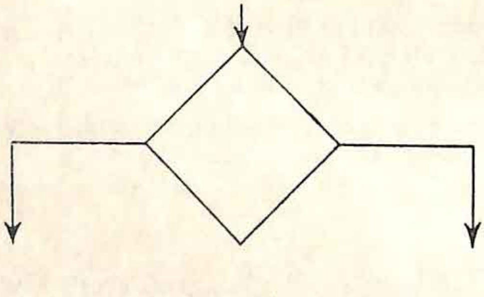
Flowchart. A flowchart is a pictorial representation of the steps involved in the step-by-step procedure (algorithm). The standard symbols used in flowcharting are :



Rectangles used to describe all processing operations performed by a computer such as calculations or assignment.

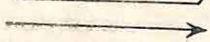


Parallelograms used to indicate computer input or output.



Diamonds used to indicate a decision that will elicit a "Yes" or "No" answer.

Ovals used to denote beginning or end of a program.



Arrows used to indicate the direction of flow in the flowchart.

In your study of mathematics up till now you have solved many problems relating to profit and loss, ratio and proportion, simple and compound interest, average calculation etc. The methods of problem solving which you used for those problems can be nicely and accurately represented by listing the steps or by drawing a flow chart. The following examples will illustrate this.

Example 1. Suppose a boy has got his marks in 3 subjects. The following flowchart calculates the average marks and outputs Roll no. and average marks.

Algorithm.

Step 1. Read Roll No. and marks in 3 subjects and call them R, A, B and C respectively.

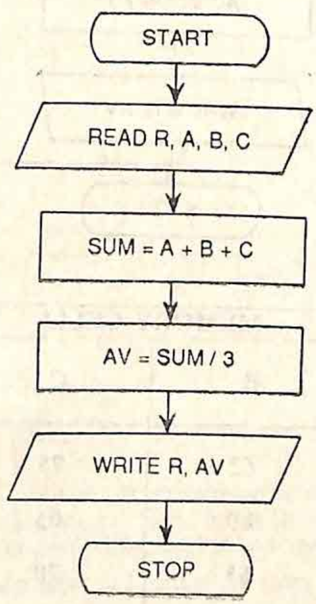
Step 2. Add A, B, C and call it SUM.

Step 3. Divide SUM by 3 and call it AV.

Step 4. Write R and AV.

Step 5. Stop.

The flowchart for the above procedure is as follows :



Now suppose we wish to find the average marks of all the students in the class. What do we do ? Do we write the procedure as many times ? This is hardly an efficient way.

Since all the steps have to be repeated for many students, we simply modify the above flowchart as shown on page 194.

Here we have introduced an arrow after the WRITE step which goes to the beginning. Following this arrow we can repeat the same steps many times. Each time the data in A, B and C will be different.

Suppose this procedure is to be followed for 3 students with following marks :

50, 62, 75 marks for first student.

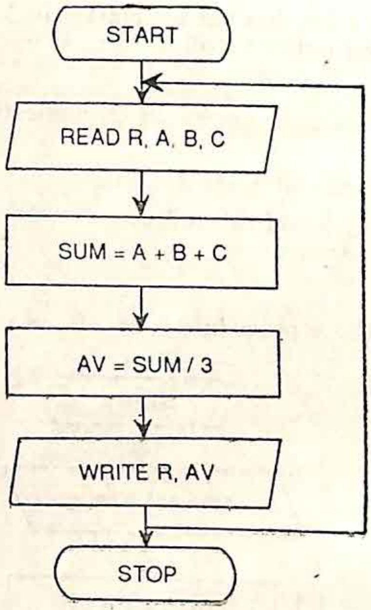
55, 80, 65 marks for second student.

75, 65, 70 marks for third student.

Let's do tracing of the flow chart with the above values.

When we come to READ box for the first time 50, 62, 75 will be stored in cells A, B, C. These values will be added and the result 187 will be stored in SUM. Average will be

calculated and printed. Now instead of stopping, the new arrow takes us back to the read box. This is called **looping**. New values are read and stored in locations A, B, C overwriting the previous values. New result taking the values 55, 80, 65 will be stored in SUM. With that the previous values are lost. The change in values of A, B, C, SUM and AV can be depicted as follows :



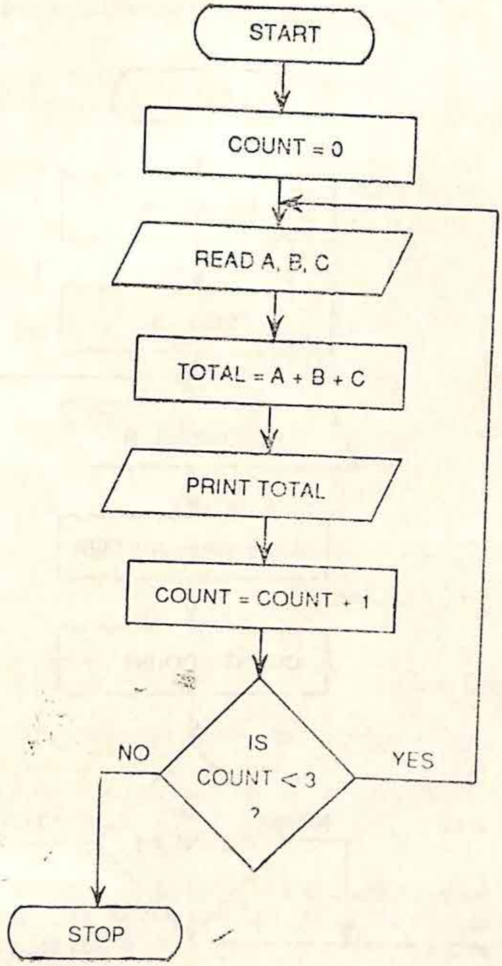
TRACING :

	MEMORY CELLS				
ITERATION	A	B	C	SUM	AV
1	50	62	75	187	62·33
2	55	80	65	200	66·66
3	75	65	70	210	70

In this way, by using the same memory locations and same step of procedures, we can do the processing for a number of students. In the above process the control moves back to box no. 2 even after processing is over for all the three students. Control never comes to stop box. Therefore flow chart has to be modified such that the control moves back only three times. Here comes the **counting concept**.

Consider the flow chart on page 195.

In this flowchart one more variable COUNT has been used. This is used for counting how many times the process is repeated. In the beginning COUNT is assigned starting value zero. Then all the processing steps follow. Before going back to the first step, COUNT is increased by one *i.e.*, new value is stored in COUNT. Now COUNT=previous value of COUNT+1. In the beginning COUNT was equal to 0. So after processing for first student is over, COUNT will become 1. Then the value of COUNT is compared with 3 and as 1 is less than 3, the condition is true, so control moves back to step no. 3. The steps 3, 4, 5 and 6 are done again and in step 6 COUNT gets increased by 1, *i.e.*, the value of COUNT is now 2. Again the condition is tested and control goes back to step 3. As soon as processing for all the three students is over *i.e.*, the COUNT becomes equal to 3. The condition whether COUNT is



less than 3 or not is checked again. Since this condition is no longer true, the control is shifted to NO option and to STOP box. The processing comes to an end. Thus we are able to count the number of times the flowchart is performed and stop it at exactly the right point.

Example 3. Draw a flow chart for finding the sum of 5 given numbers.

Let us first work out its step-by-step procedure. Since we have to read numbers more than once we require to use the counter.

Step 1. Set COUNT equal to zero. (Since we have not started counting.)

Step 2. Set SUM equal to zero. (Since we have not started adding numbers.)

Step 3. Read the number.

Step 4. Add number to SUM and call it SUM.

Step 5. Add 1 to COUNT and call it COUNT.

Step 6. If COUNT $<$ 5 then go to step 3 otherwise go to step 7.

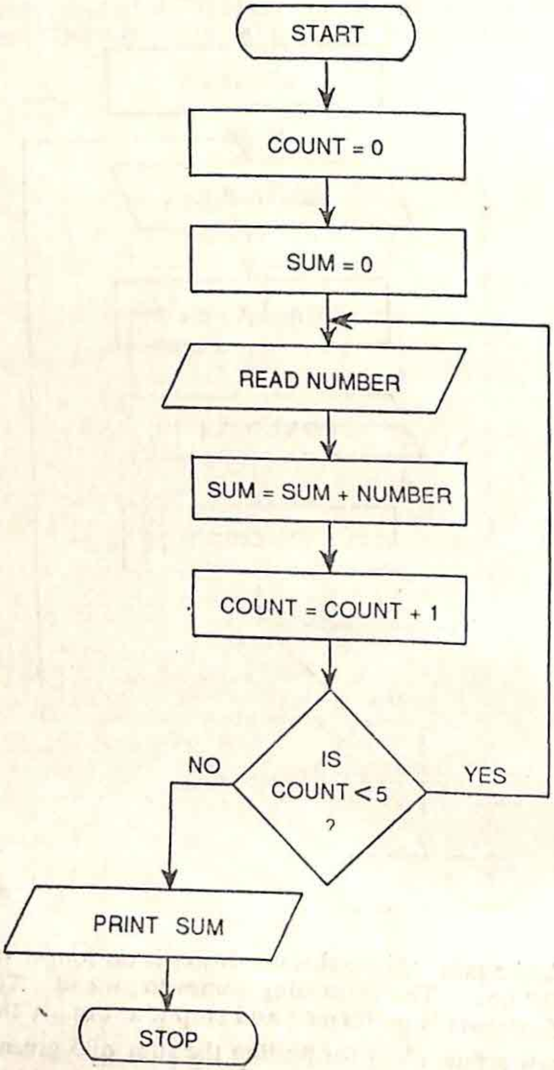
Step 7. Write SUM.

Step 8. Stop.

In above algorithm we will keep on reading the numbers till all the 5 numbers are read *i.e.*, COUNT=5. In step 7 the SUM is printed.

This can be represented in the flow chart form as given below ;

Flow Chart



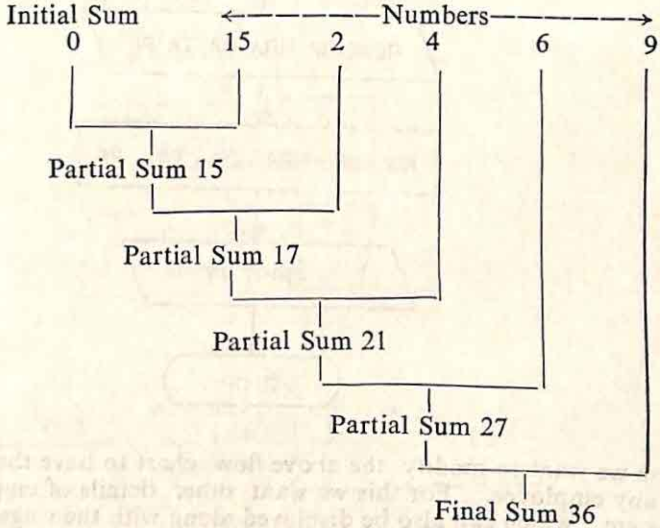
Now let us do tracing to see how the values change in the memory cells during processing. In this problem three memory cells are used. COUNT and SUM have starting value zero. Another variable NUMBER is used for holding the values.

Let's do tracing using 5 numbers : 15, 2, 4, 6, 9.

Number (*Read Number*)	*Sum* (*Sum=Sum+Count*)	*Count* (*Count=Count+1*)	*Condition* *Count<5*	*Result* *Print Sum*
	0	0		
15	0+15=15	0+1=1	Yes	
2	15+2=17	1+1=2	Yes	
4	17+4=21	2+1=3	Yes	
6	21+6=27	3+1=4	Yes	
9	27+9=36	4+1=5	No	36

When the loop ends, the SUM cell in the memory has value 36 which gets displayed on the screen.

We find from tracing, SUM cell holds partial sum of numbers. Numbers are added one by one. They are held temporarily as partial sum till all the numbers are added. This can be explained as :



Let's take some more examples to clarify the concept of flow charts.

Example 4. Calculate the Net Salary of an employee if Basic Pay, House Rent Allowance (HRA), Dearness Allowance (DA), Travelling Allowance (TA) and Provident Fund are given.

In this particular problem we have to calculate the net salary of one employee. We know the Basic Salary, House Rent Allowance, Travelling Allowance, Dearness Allowance and Provident Fund. We also know that H.R.A., T.A. and D.A. have to be added to the Basic Pay to get the Gross Salary. Provident Fund has to be deducted from Gross Salary to get the Net Salary of the employee. Now let's set the variables ; BP will stand for Basic Pay, HRA for House Rent Allowance, TA for Travelling Allowance and PF for Provident Fund. So Net Salary (NS) will be calculated using the following formula :

$$NS=BP+HRA+DA+TA-PF$$

Short Names for Memory Cells

BP, HRA : Basic Pay, House Rent Allowance

DA, TA : Dearness Allowance, Travelling Allowance

PF : Provident Fund

NS : Net Salary

Step-by-step Procedure

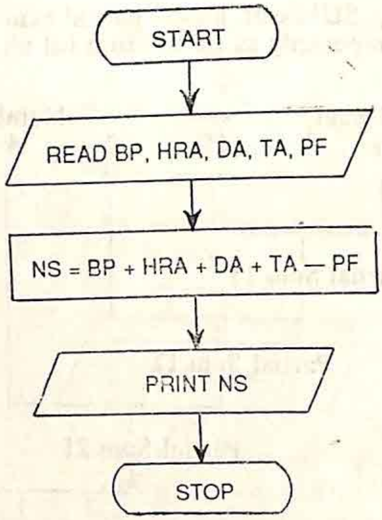
Step 1. Read the values of BP, HRA, DA, TA and PF.

Step 2. Calculate wage using the formula NS=BP+HRA+DA+TA−PF

Step 3. Print NS.

Step 4. Stop.

Flow Chart



Suppose we want to modify the above flow chart to have the possibility of calculating the wage of any employee. For this we want other details of employees like employee no., employee name etc., which can also be displayed along with the wage of that employee. Now we would like to introduce the loop and loop check. We want to repeat the same module to calculate the pay as many times as required and stop the loop using some check to see if salary of all the employees has been calculated. In other words if there are still more employees we should continue with the loop, otherwise we should come to stop.

Short Names for Memory Cells

EMP NO. : Employee number

EMP NAME : Employee name

Other variable names are same as in the previous example.

Step-by-step Procedure

Step 1. Input the values of EMP NO., EMP NAME, BP, HRA, DA, TA, PF.

Step 2. Calculate net salary using formula NS=BP+HRA+TA+DA−PF.

Step 3. Print EMP NO., EMP NAME and NS.

Step 4. Check if there are any more employees.

Step 5. If Yes go to step 1 else go to step 6.

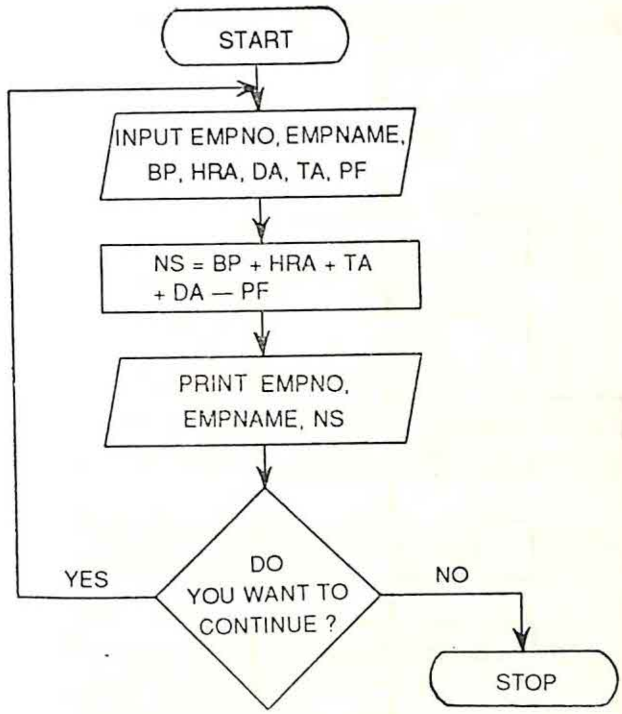
Step 6. Stop.

The flow chart for the above example is given on page 199.

Example 5. To find out the average of physics marks of students in a given class. In this example we will follow the same procedure as we do in Algebra.

Suppose there are 10 students in the class and we want to calculate the average of their physics marks. We simply add the ten marks and divide the sum by 10. Here we know that total students in the class are 10. However their number can vary. So we should know how many students are there. Let us call the number of students as NS. We can read the physics marks of each student one by one until the number of students matches with NS. We can also keep on adding the physics marks in the SUM. Once we come out of the loop, we can divide the SUM by NS to get the average.

Flow Chart



Short Names for Memory Cells

NS : Number of students

PM : Physics marks of any one student

SUM : Partial sum

COUNT : Counter for students

AVE : Average physics marks.

Step-by-step Procedure

Step 1. Input NS : the number of students in the class.

Step 2. Initialize the value of SUM as 0.

Step 3. Initialize the value of COUNT as 0.

Step 4. Input the value of PM : Physics marks of next student.

Step 5. Add PM to SUM and call it SUM.

Step 6. Add 1 to COUNT and call it COUNT.

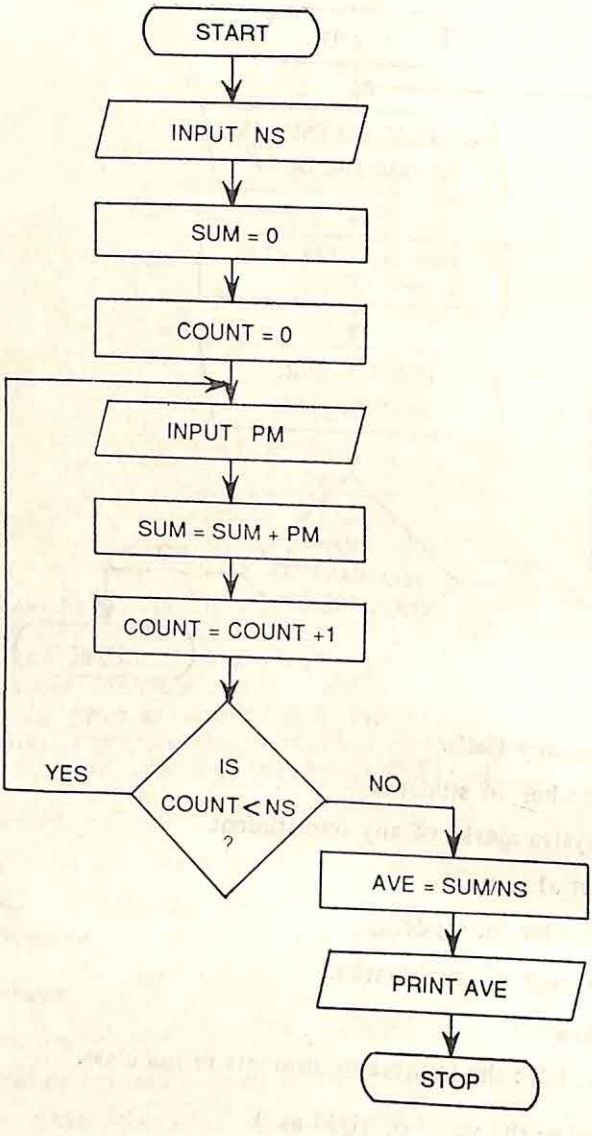
Step 7. If COUNT is less than NS then go to step 4 else go to step 8.

Step 8. Divide SUM by NS to give AVE.

Step 9. Write AVE.

Step 10. Stop.

Flow Chart



The above flow chart can be divided into three parts : Initialization, processing and printing of result. The first four boxes constitute the initialization part. Here the initial values of different names such as NS, SUM and COUNT are set. Boxes 5 to 8 constitute the processing part. Here there is one loop consisting of steps 4 to 7. In this loop, we read the physics marks of each student, calculate the running sum, keep track of number of students processed so far and test to see if marks of all students have been processed. In box 9, we calculate the average. In box 10, the result is printed. This is very typical of flow chart and many flow charts can be seen as consisting of these parts.

Example 6. To calculate simple interest of 25 customers, where Principal, Rate of Interest and Time Period are given to you.

To calculate simple interest, we need to know the values of principal, rate of interest and time period. We should have 25 sets of these values because these values will vary from

customer to customer. We can not give all the values together. We will take one set at a time, calculate interest using the formula, $I = P * R * T/100$ and print the value of interest. We repeat this loop 25 times. For this we should have counter and counter check. The procedure will be like this :

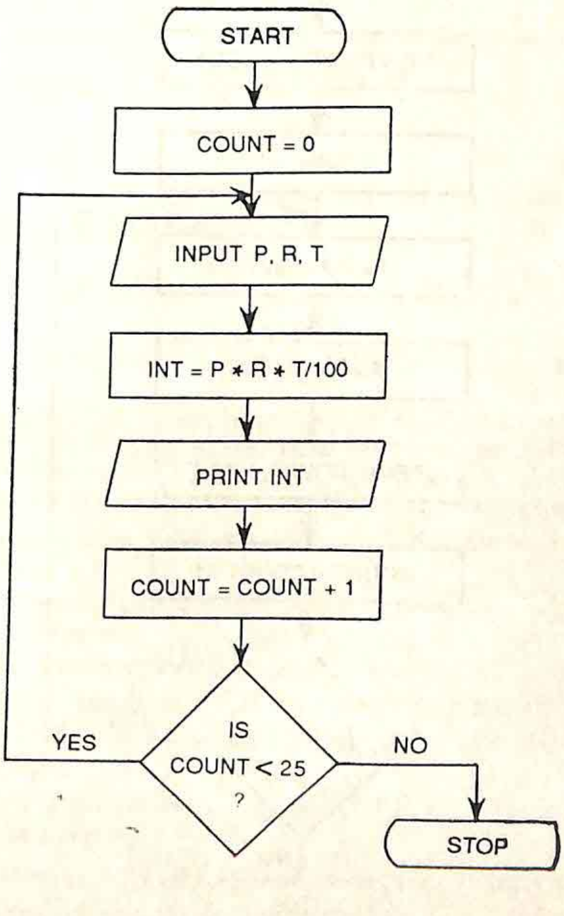
Short Names for Memory Cells

P : Principal
R : Rate of interest
T : Time period
INT : Interest to be calculated
COUNT : For counting the number of customers

Step-by-step Procedure

Step 1. Initialize COUNT as zero.
Step 2. Input the values of Principal, Rate of interest and Time period.
Step 3. Calculate Interest using the formula INT=P*R*T/100.
Step 4. Print INT.
Step 5. Add 1 to COUNT and store the new value in COUNT again.
Step 6. If COUNT<25 then go to step 2 else go to step 7.
Step 7. Stop.

Flow Chart

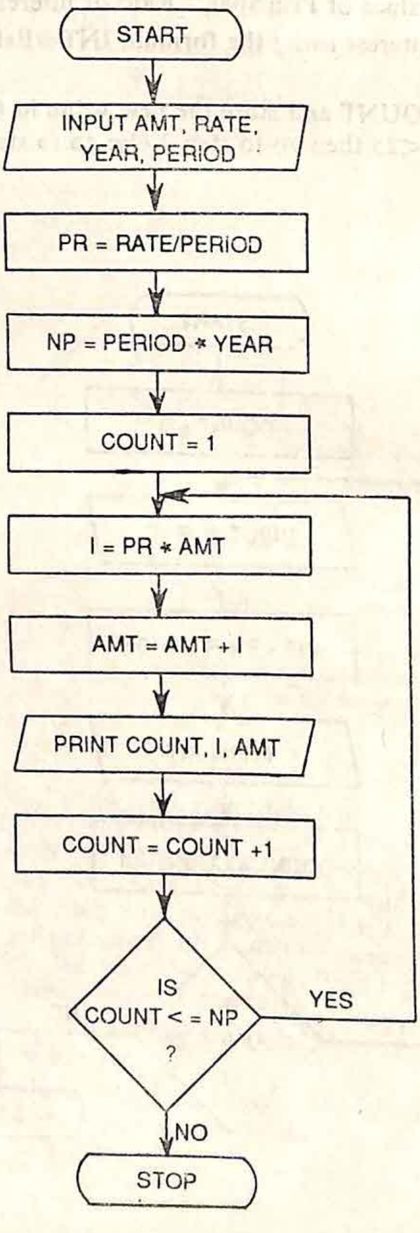


Example 7 : Sudhir deposited Rs. 20,000 in his Savings Bank Account three years ago. Rate of interest is 12% per annum and is to be compounded monthly. Print table of the current amount in Sudhir's account each month for three years.

Short Names for Memory Cells

RATE	: Rate of interest
AMT	: Amount
T	: Time period
I	: Interest
COUNT	: Counter
PR	: Period rate

Flow Chart



The flowchart given on page 202 is based on the following algorithm :

Algorithm

Step 1. Input the AMOUNT, RATE, no. of periods per year.
Step 2. Divide RATE by periods per year and call it PR (period rate).
Step 3. Multiply periods per year with number of years and call it NP (number of periods).
Step 4. Initialize the counter to 1.
Step 5. Multiply AMT by PR and call it I.
Step 6. Add I to AMT and call it AMT.
Step 7. Write COUNT, I, AMT.
Step 8. Add I to COUNT and call it COUNT.
Step 9. If COUNT is less than or equal to the number of periods (NP) then go to Step 5 otherwise go to Step 10.
Step 10. Stop.

Example 8. Calculate the HCF (Highest Common Factor) for two given numbers.

Solution

Suppose two numbers are 27 and 45. Compare the two numbers and treat the bigger of the two numbers as numerator and the other number as denominator. In this case 27 is denominator and 45 is numerator.

Step 1. Divide numerator by denominator.

```
(denominator)      27)   45   (1 (quotient)
                         27
                         --
(remainder)              18
```

Step 2. Divide the divisor of Step 1 by the remainder in Step 1.

```
                   18) 27 (numerator) (1
                       18
                       --
(Remainder)             9
```

Step 3. Repeat Step 2 with new values of divisor and remainder obtained in Step 2 until remainder becomes zero.

```
                   19)  18  (2
                        18
                        --
(Remainder)              0
                        --
```

The last divisor is highest common factor. ∴ HCF of 27 and 45 is 9. In Step 1 divide numerator by denominator and find the remainder. If the remainder is equal to zero then denominator is the answer otherwise make your denominator the numerator and remainder the denominator and again repeat the process till you have remainder equal to zero.

In above example, in Step 3 the remainder comes to be zero. So, the denominator *i.e.*, 9 is the HCF of 27 and 45.

Now let us develop an algorithm for the above problem.

Algorithm. To find HCF of two given numbers A and B.

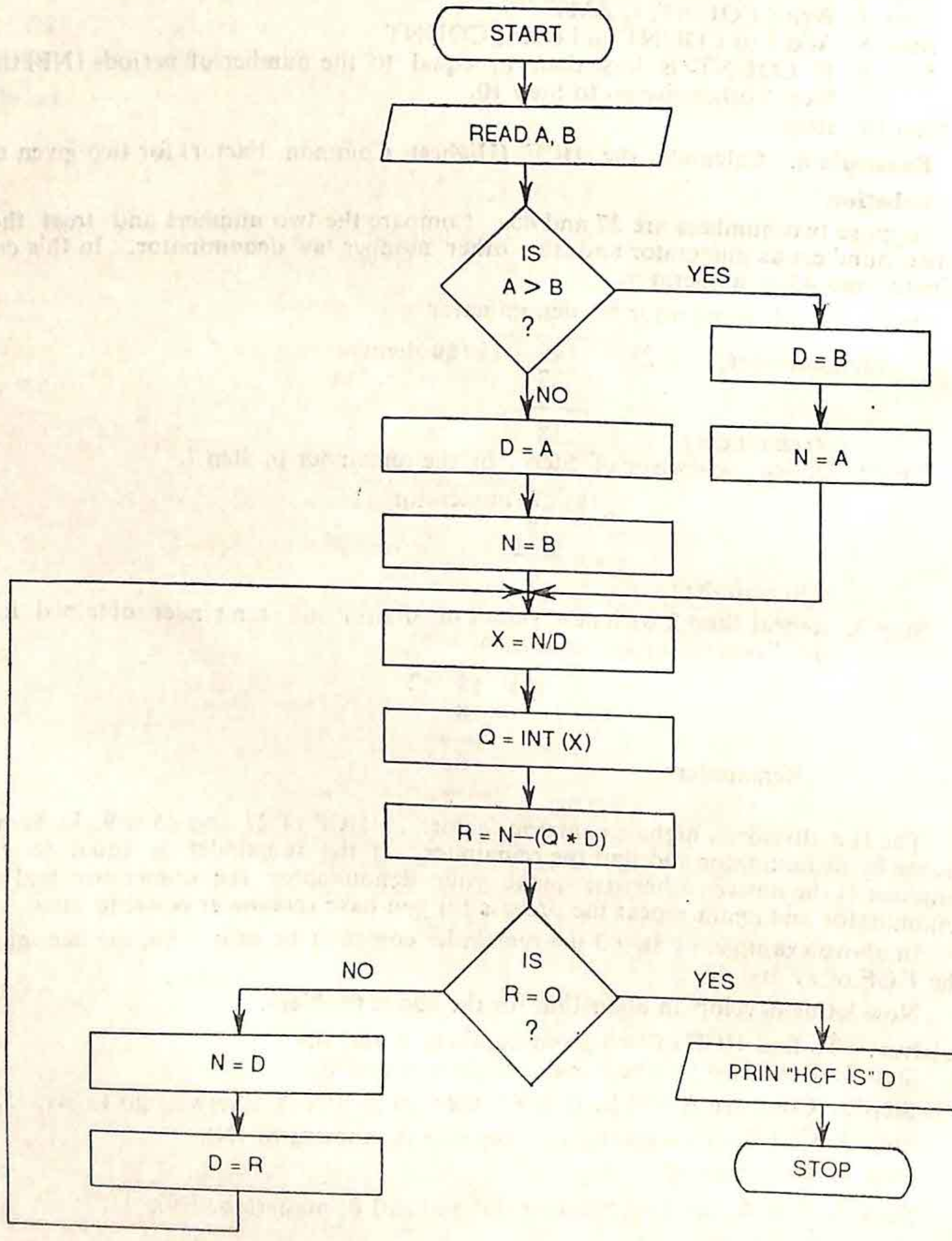
Step 1. Read two numbers and call them A and B.
Step 2. Compare A and B. If $B > A$ then go to Step 5 otherwise go to Step 3.
Step 3. Call B, denominator (D) and call A, numerator (N).
Step 4. Goto Step 6.
Step 5. Call A, the denominator (D) and call B, numerator (N).
Step 6. Divide N by D.
Step 7. Check if remainder $(R)=0$ then go to Step 11 otherwise go to Step 8.
Step 8. Call denominator (D) as numerator (N).

Step 9. Call remainder (R) as D.
Step 10. Go back to Step 6.
Step 11. Write "Highest Common Factor is" D.
Step 12. Stop.

Before we convert this algorithm into flow-chart Step 7 requires further expansion. The problem is how to calculate the remainder.

Step (*a*) divide N by D and call it X (Quotient).
Step (*b*) take integer value of X [there is INT function available] and call it Q.
Step (*c*) Subtract (Q * D) from N and call it R.

Now we can draw the flow-chart for finding HCF of 2 given numbers A and B.



Let's do tracing of the above flow chart taking the two numbers as 27 and 45.

Tracing

A	B	D	N	X	Q=INT (X)	R=N−(Q * D)	Result
27	45	27	45	1.6	1	18	
		18	27	1.5	1	9	
		9	18	2	2	0	9

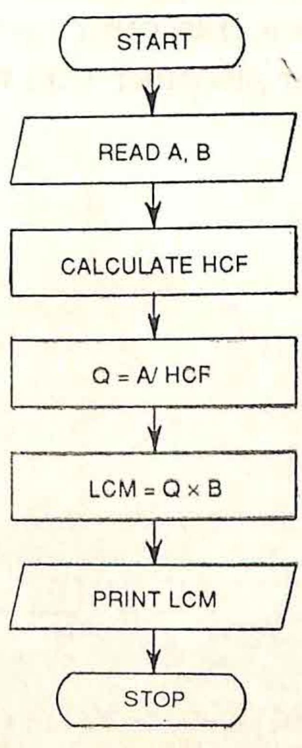
Example 9. Calculate the LCM of two given numbers A and B.

Solution. For finding the LCM of two numbers, divide one of the numbers by their HCF and multiply the quotient by the other number *i.e.*,

$$LCM=\frac{A}{HCF} * B$$

Algorithm :

Step 1. Read two numbers and call them A and B.
Step 2. Calculate their HCF (following the procedure given in previous example).
Step 3. Divide A by HCF and call it Quotient.
Step 4. Multiply Quotient by B and call it LCM.
Step 5. Write LCM.
Step 6. Stop.



HDF is calculated following the same procedure as explained in the previous example.

Example 10. A person buys a number of goods from a departmental store in a sale. The departmental store is offering discount of 12% on each item. Additional discount of 5% is given if the person has made purchases in excess of Rs. 1000/-. Input consists of number of items purchased and cost price of each item. Calculate the discount and sale price.

Algorithm

Step 1. Read the number of items purchased and call it N.

Step 2. Initialize the item counter (I COUNT) as zero.

Step 3. Initialize TOTAL as zero.

Step 4. Read the cost of item and call it COST.

Step 5. Add COST to TOTAL and call it TOTAL.

Step 6. Add 1 to I COUNT and call it I COUNT.

Step 7. If I COUNT $<$ N then go to Step 4.

Step 8. Multiply TOTAL by ·12 and call it DISCOUNT 1.

Step 9. Subtract DISCOUNT 1 from TOTAL and call it SALE PRICE.

Step 10. If TOTAL is $<$ 1000 then DISCOUNT 2=0 and go to step 13 otherwise go to Step 11.

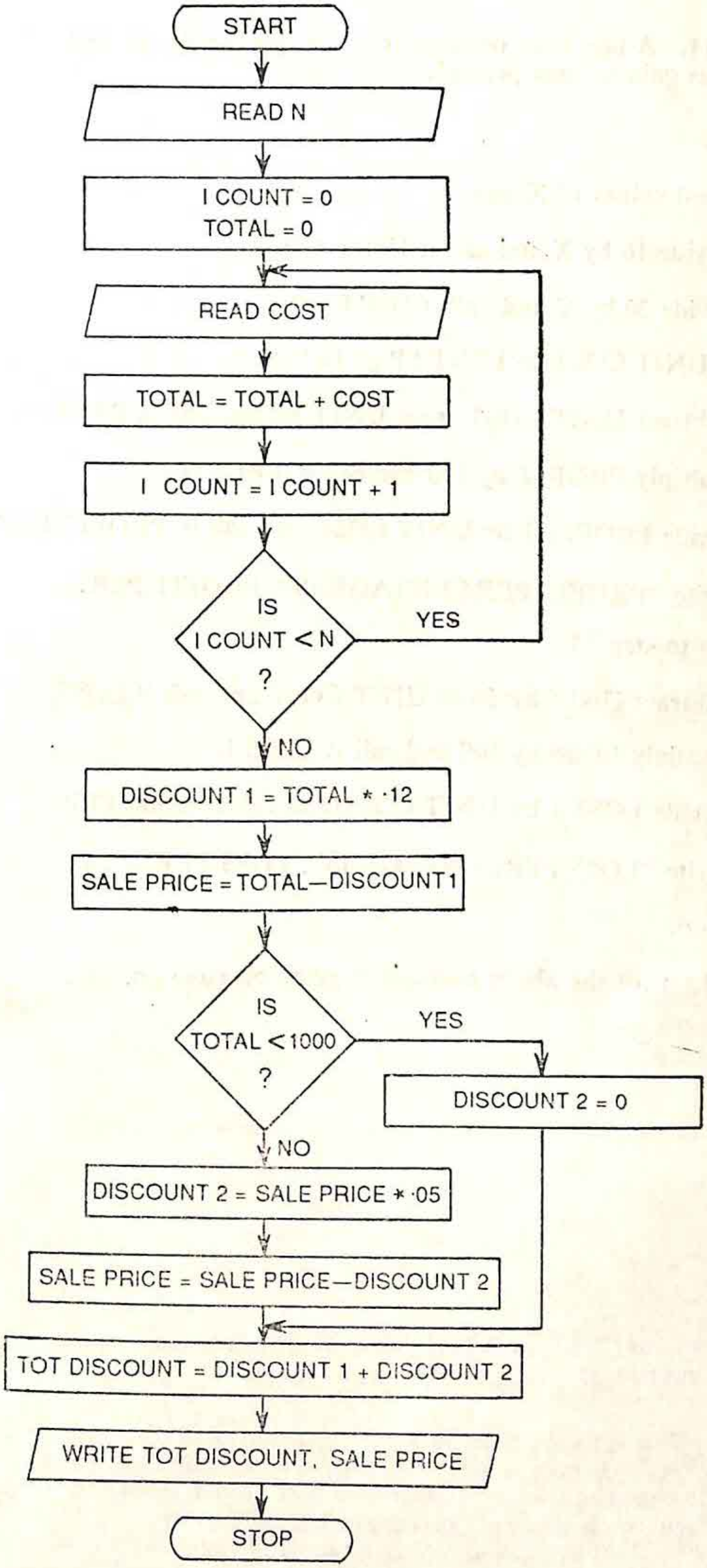
Step 11. Multiply SALE PRICE by ·05 and call it DISCOUNT 2.

Step 12. Subtract DISCOUNT 2 from SALE PRICE and call it SALE PRICE.

Step 13. Add DISCOUNT 1 and DISCOUNT 2 and call it TOT DISCOUNT.

Step 14. Write TOTAL, TOT DISCOUNT, SALE PRICE.

Step 15. Stop.


START
READ N
I COUNT = 0
TOTAL = 0
READ COST
TOTAL = TOTAL + COST
I COUNT = I COUNT + 1
IS
I COUNT < N
?
YES
NO
DISCOUNT 1 = TOTAL * ·12
SALE PRICE = TOTAL—DISCOUNT1
IS
TOTAL < 1000
?
YES
DISCOUNT 2 = 0
NO
DISCOUNT 2 = SALE PRICE * ·05
SALE PRICE = SALE PRICE—DISCOUNT 2
TOT DISCOUNT = DISCOUNT 1 + DISCOUNT 2
WRITE TOT DISCOUNT, SALE PRICE
STOP

Example 11. A boy buys oranges at X oranges for Rs. 16 and sells them at Y oranges for Rs. 20. Find his gain or loss percent.

Algorithm

Step 1. Read values of X and Y.

Step 2. Divide 16 by X and call it UNIT COST.

Step 3. Divide 20 by Y and call it UNIT SP.

Step 4. If UNIT COST > UNIT SP go to Step 9.

Step 5. Subtract UNIT COST from UNIT SP and call it PROFIT.

Step 6. Multiply PROFIT by 100 and call it PROFIT 1.

Step 7. Divide PROFIT 1 by UNIT COST and call it PROFIT PER.

Step 8. Write "PROFIT PERCENTAGE IS", PROFIT PER.

Step 9. Go to step 13.

Step 10. Subtract UNIT SP from UNIT COST and call it LOSS.

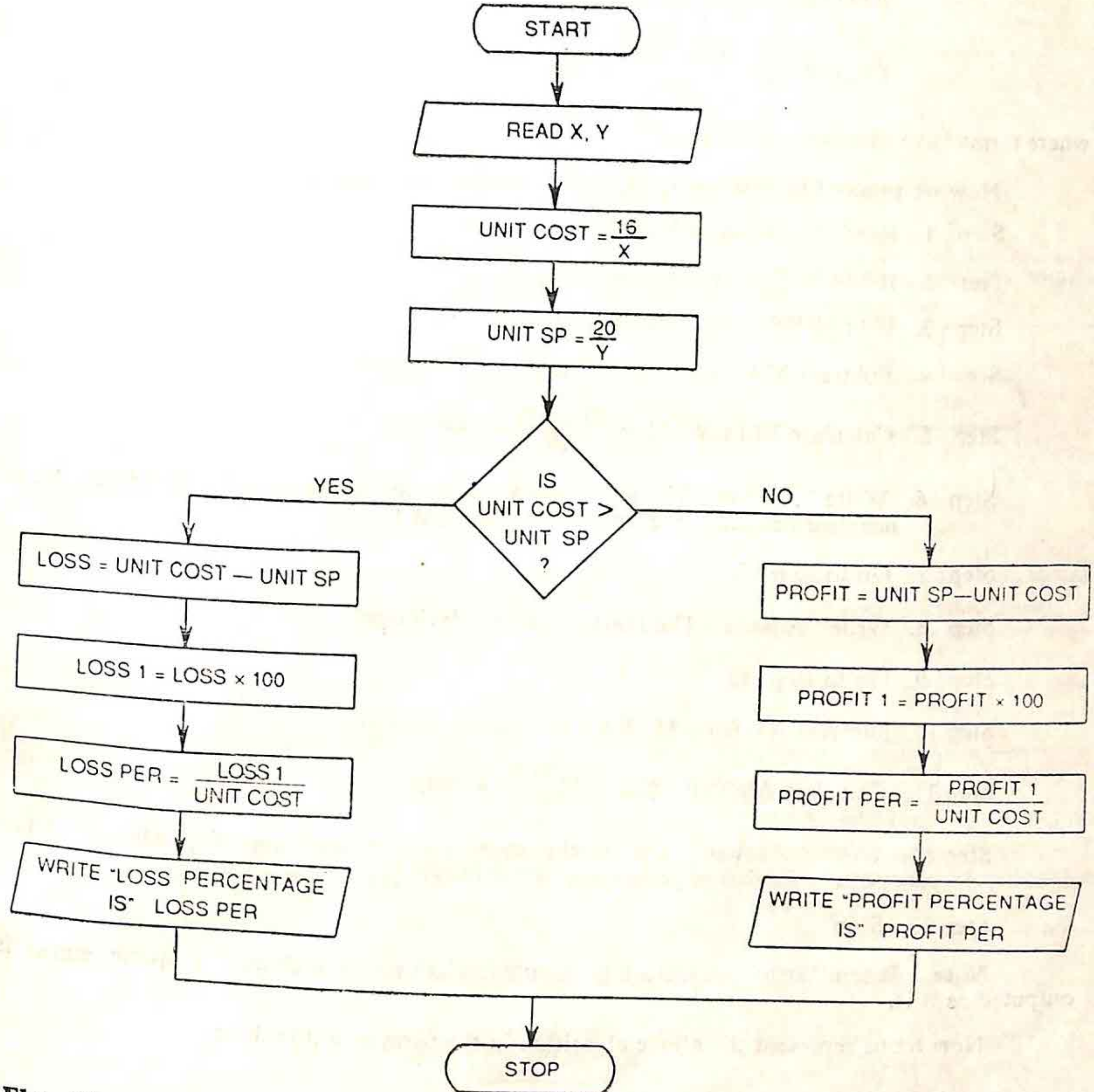
Step 11. Multiply LOSS by 100 and call it LOSS 1.

Step 12. Divide LOSS 1 by UNIT COST and call it LOSS PER.

Step 13. Write "LOSS PERCENTAGE IS", LOSS PER.

Step 14. Stop.

The flow chart for the above example is given on page no. 209.



Flow Chart

Example 12. Prepare an algorithm to find the deviation from the average marks (positive or negative) and corresponding percentage of deviations given the average marks and actua marks obtained.

Solution. Before proceeding to develop an algorithm for the above problem let us understand the terms clearly. A candidate is considered above average (positive deviation) if his marks are more than the average marks and is considered below average (negative deviation) if his marks are less than the average marks. If his marks are equal to average marks he is considered to be average (no deviation). Various formulas which we shall be using are :

ABOVE=MARKS—AV

BELOW=AV—MARKS

$$\text{ABOVE }\% = \frac{\text{ABOVE}}{\text{AV}} * 100$$

$$\text{BELOW }\% = \frac{\text{BELOW}}{\text{AV}} * 100$$

where terms have obvious meanings.

Now we proceed to develop an algorithm for the above problem.

Step 1. Read the values of MARKS and AV.

Step 2. If MARKS=AV then go to step 8.

Step 3. If MARKS>AV then go to step 10.

Step 4. Subtract MARKS from AV and call it BELOW.

Step 5. Calculate BELOW $\% = \frac{\text{BELOW}}{\text{AV}} * 100$.

Step 6. Write "Answer : The marks show negative deviation of", BELOW ; "The negative deviation percentage is", BELOW %.

Step 7. Go to step 13.

Step 8. Write "Answer : The marks show no deviation".

Step 9. Go to step 13.

Step 10. Subtract AV from MARKS and call it ABOVE.

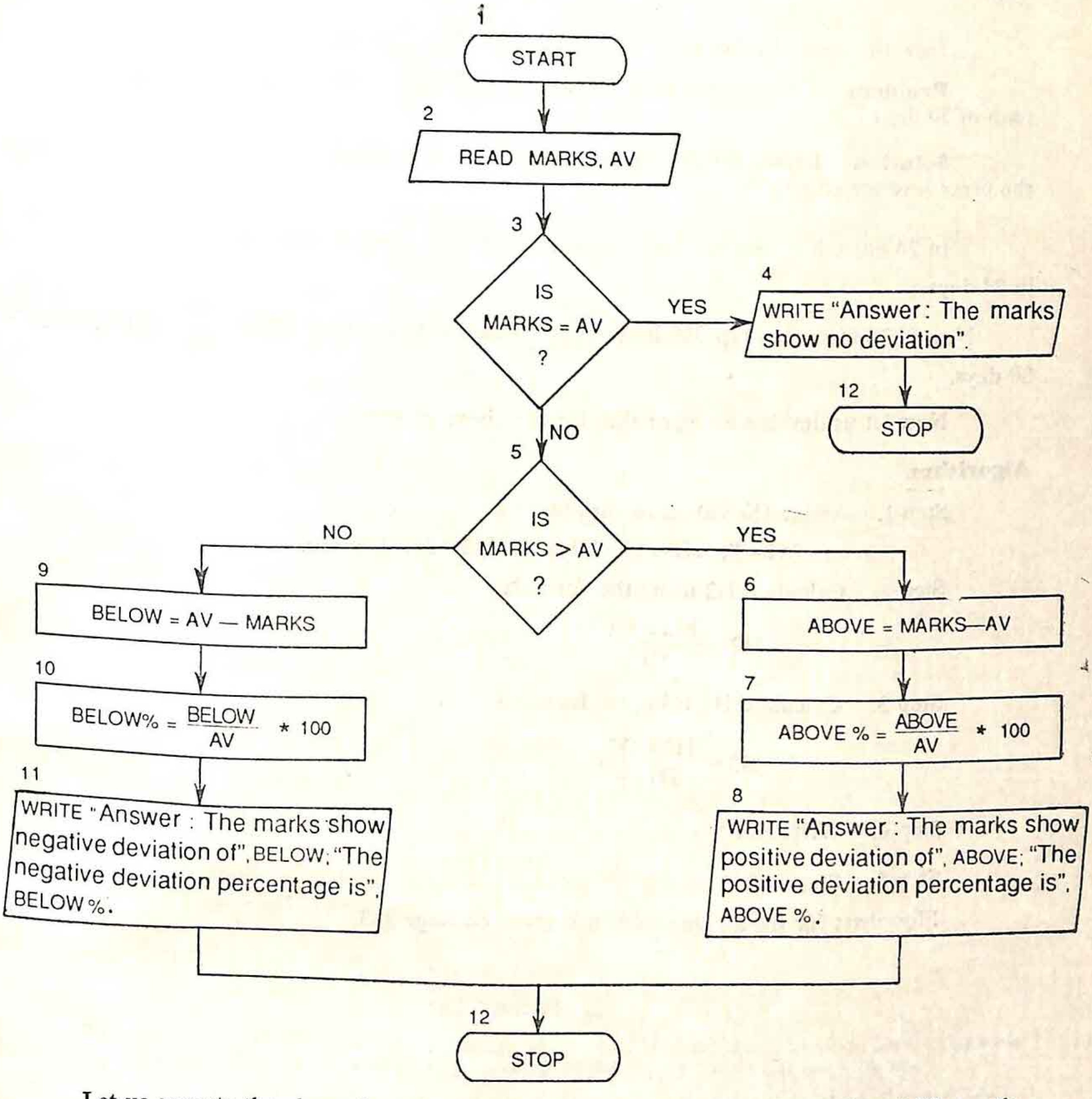
Step 11. Calculate ABOVE $\% = \frac{\text{ABOVE}}{\text{AV}} * 100$.

Step 12. Write "Answer : The marks show positive deviation of", ABOVE ; "The positive deviation percentage is", ABOVE %.

Step 13. Stop.

Note. When 'Write' instruction is executed, whatever is included in quote marks is outputed as it is.

Now let us represent the above algorithm in the form of a flowchart.



Let us execute the above flow chart taking 75 as the value of MARKS and 52 as the value of AV.

Execution beging at START box numbered 1. When we reach READ box the values of MARKS and AV are read. Now MARKS=75 and AV=52.

In box numbered 3 the values of MARKS and AV are compared. Since MARKS is not equal to AV we come to box numbered 5. Here again their values are compared. Since MARKS is greater than AV control takes us to box numbered 6.

Here value of ABOVE is calculated. ABOVE=75−52=23. Next we come to box numbered 7. Here we calculate ABOVE %. ABOVE $\% = \frac{23}{52} \times 100 = 44{\cdot}23\%$.

Now we come to WRITE box. We write Answer : The marks show positive deviation of 23. The positive deviation percentage is 44·23%.

Now the control takes us to box numbered 12 and we stop.

Problem. If 8 men can reap 80 hectares in 24 days, how many hectares can 36 men reap in 30 days ?

Solution. Before developing an algorithm for the above problem. Let us understand the procedure for solving it.

In 24 days, if 8 men can reap 80 hectares then 36 men can reap $36\times\frac{80}{8}=360$ hectares in 24 days.

Now if 36 men can reap 360 hectares in 24 days they can reap $360\times\frac{30}{24}=450$ hectares in 30 days.

Now let us develop an algorithm for the above problem.

Algorithm.

Step 1. Assign the values to variables :

$$M1=8,\ H1=80,\ D1=24,\ M2=36,\ D2=30.$$

Step 2. Calculate H2 using the formula

$$H2=\frac{M2\times H1}{M1}.$$

Step 3. Calculate H3 using the formula

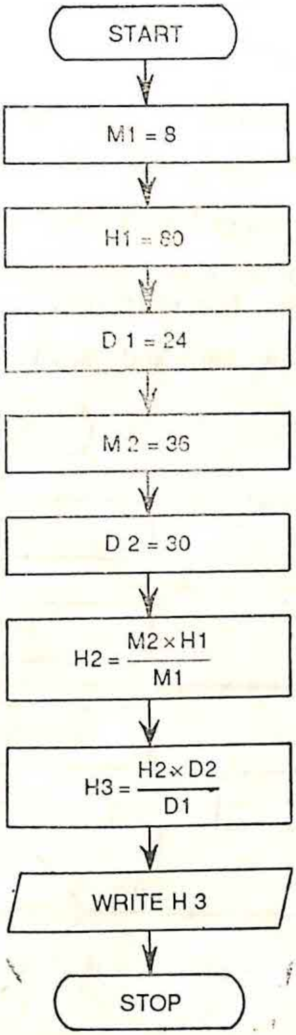
$$H3=\frac{H2\times D2}{D1}.$$

Step 4. Write H3.

Step 5. Stop.

Flowchart for the above problem is given on page 213,

Flow Chart



EXERCISE 12

1. The value of N factorial (N !) is equal to $n \times (n-1) \times (n-2) \ldots\ldots 3 \times 2 \times 1$. Draw a flowchart to calculate and print the value of N ! where N is a positive integer.
2. Make a flow chart to find the second largest number out of a given set of K numbers.
3. Draw a flow chart to find the sum of squares of first 15 numbers

$$i.e., \ 1^2+2^2+3^2+\ldots\ldots\ldots 15^2.$$

4. Draw a flow chart to print multiplication table of any number N. Write its step-by-step procedure also.
5. Draw a flow chart to compute and print grades for an examination. Input is roll no. and marks in five subjects out of 50. The grades are awarded as below :

Percentage of Marks	Grade
90 and above	'A'
80—89	'B'
70—79	'C'
Less than 70	'D'

6. Income tax T (in Rupees) is calculated on the taxable income I (in Rupees) according to the following formulae :

If $0 \leqslant I \leqslant 15{,}000$ then $T=0$

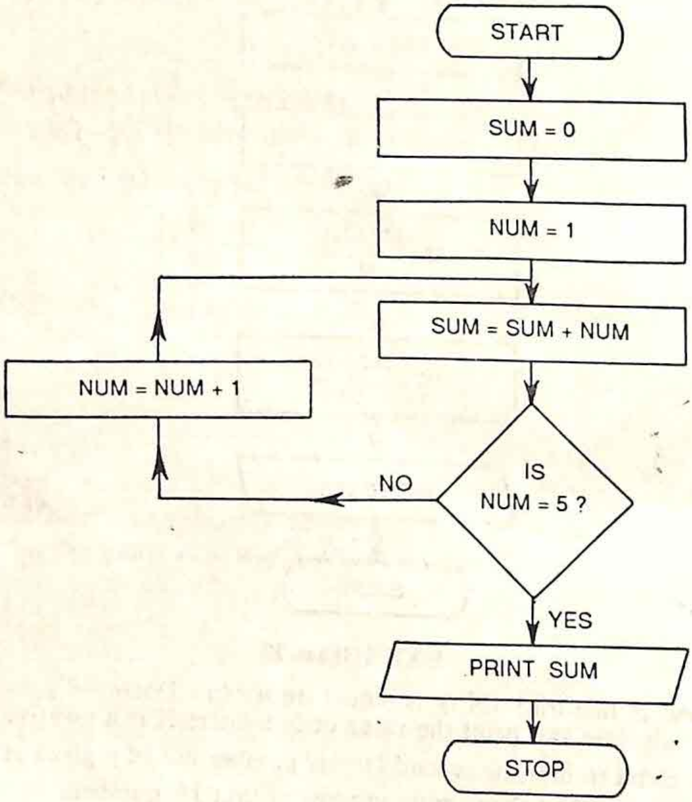
If $15{,}000 < I \leqslant 22{,}000$ then $T=15\%$ of I in excess of 15,000

If $22{,}000 < I \leqslant 45{,}000$ then $T=3000+25\%$ of I in excess of 22,000.

If $45{,}000 < I$ then $T=8{,}000+40\%$ of I in excess of 45,000.

Write an algorithm to calculate the income tax payable by a person given his taxable income. Draw the corresponding flow chart also.

7. Do tracing for the following flow charts and show the output.



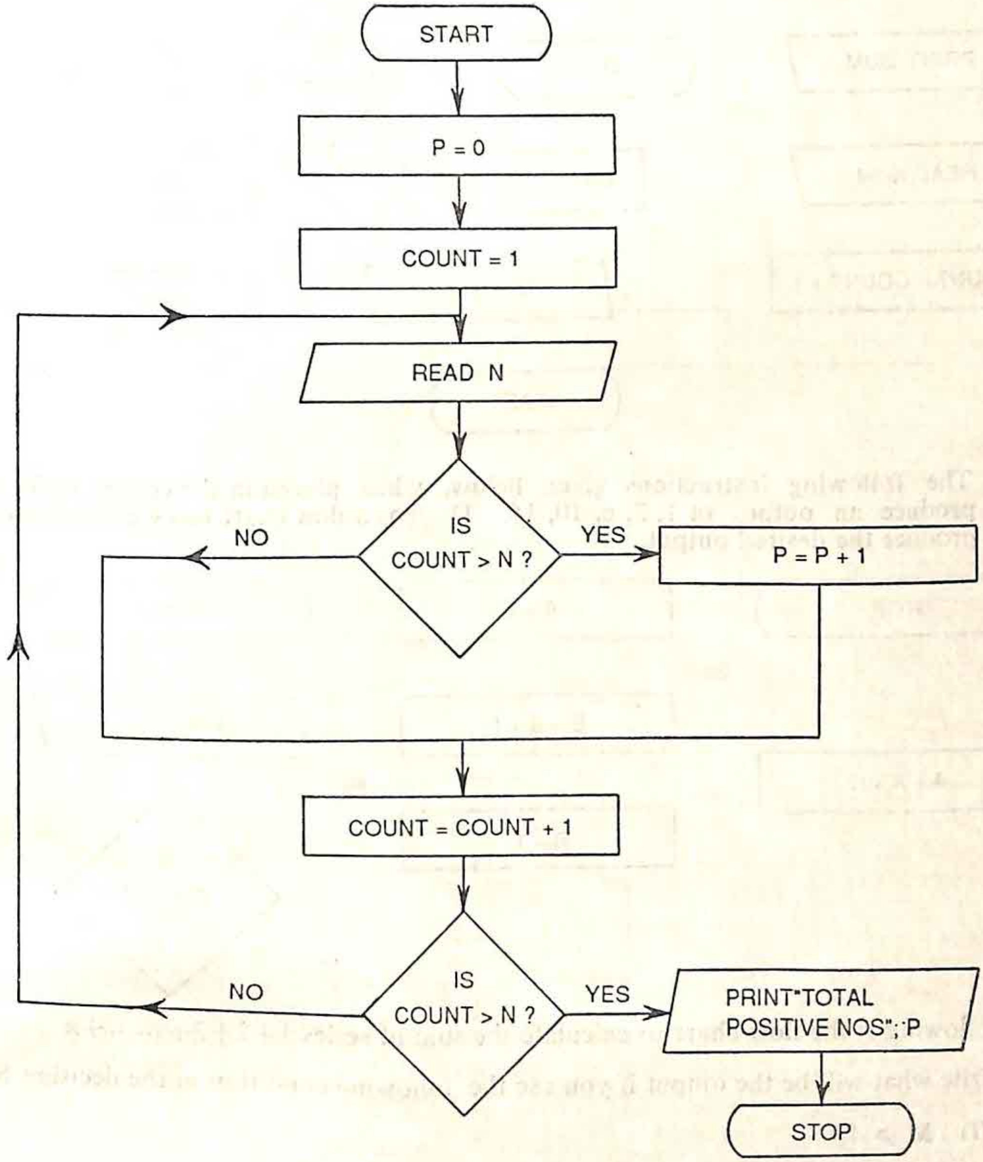
Also trace the above flow chart with following conditions :

(*i*) NUM < 5

(*ii*) NUM > 5

(*iii*) NUM >=5

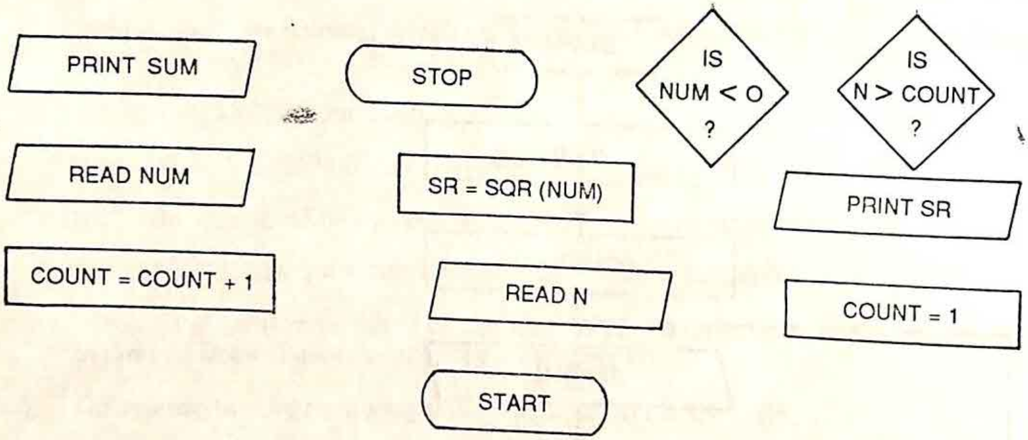
(ii)



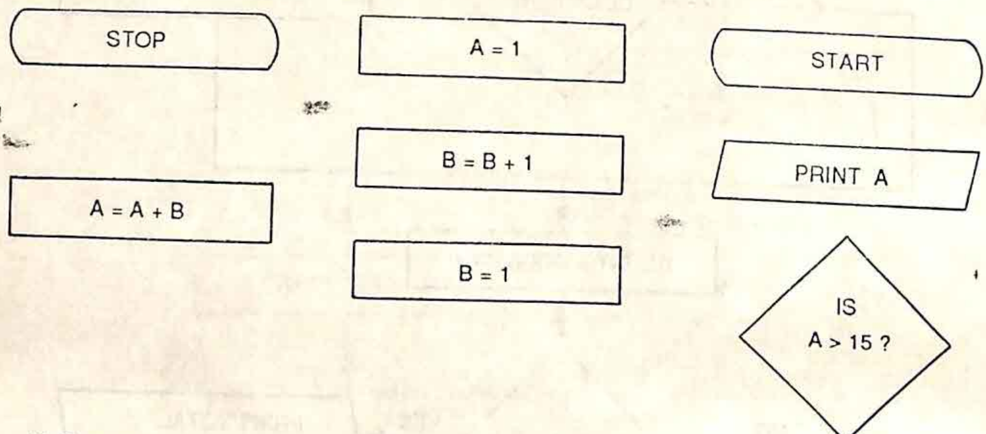
Do tracing for the above flow chart using following values and find the output :

5, 0, —62, 29, 54, 79, 48,—9,—8, 0, 59, 62

8. (*i*) Design a flow chart for finding square roots of a set of N numbers which contain few negative numbers also using the given instructions.



(*ii*) The following instructions given below, when placed in the correct order, would produce an output of 1, 3, 6, 10, 15. Design a flow chart using all instructions to produce the desired output.



9. Following is the flow chart to calculate the sum of series $1+2+3+\ldots\ldots+n$

Write what will be the output if you use the following condition in the decision box ;

(*i*) $M \geqslant 10$

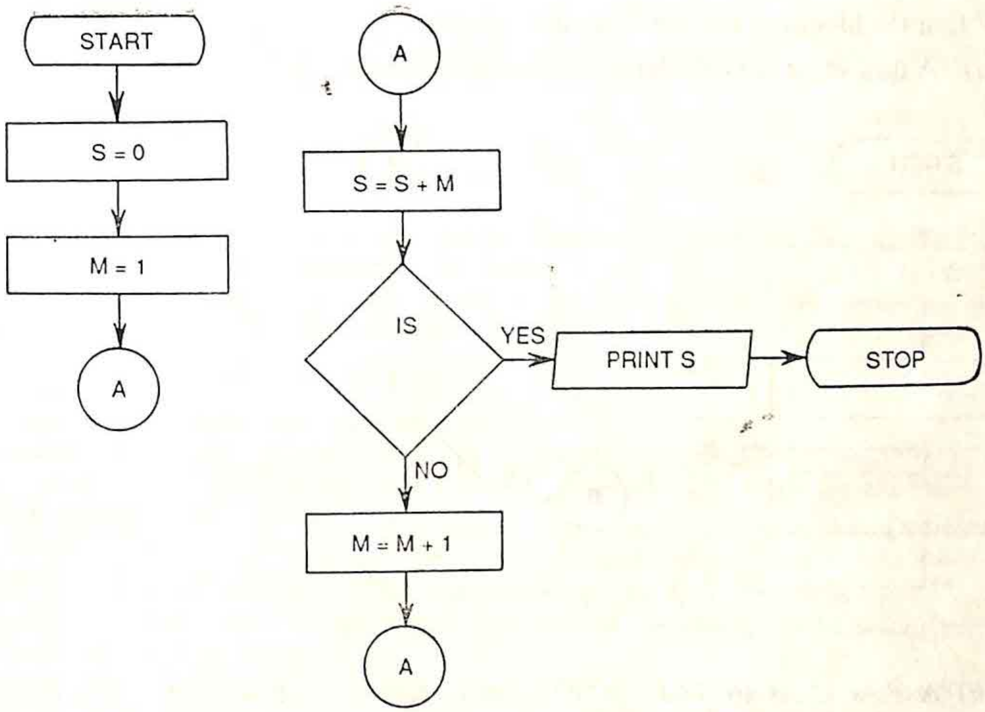
(*ii*) $M > 20$

(*iii*) $M = 0$

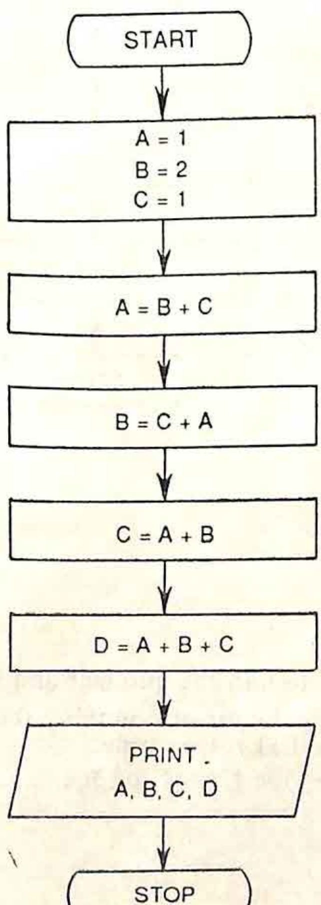
(*iv*) $M \geqslant 20$

(*v*) $M \leqslant 20$

(*vi*) $M < 20$

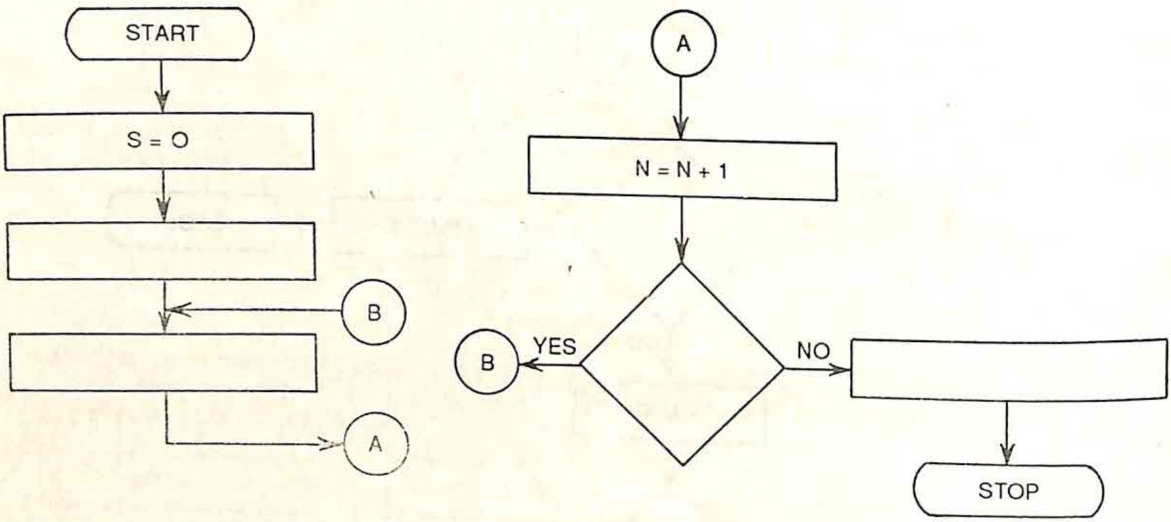


10. Give the outputs of the following flowchart.

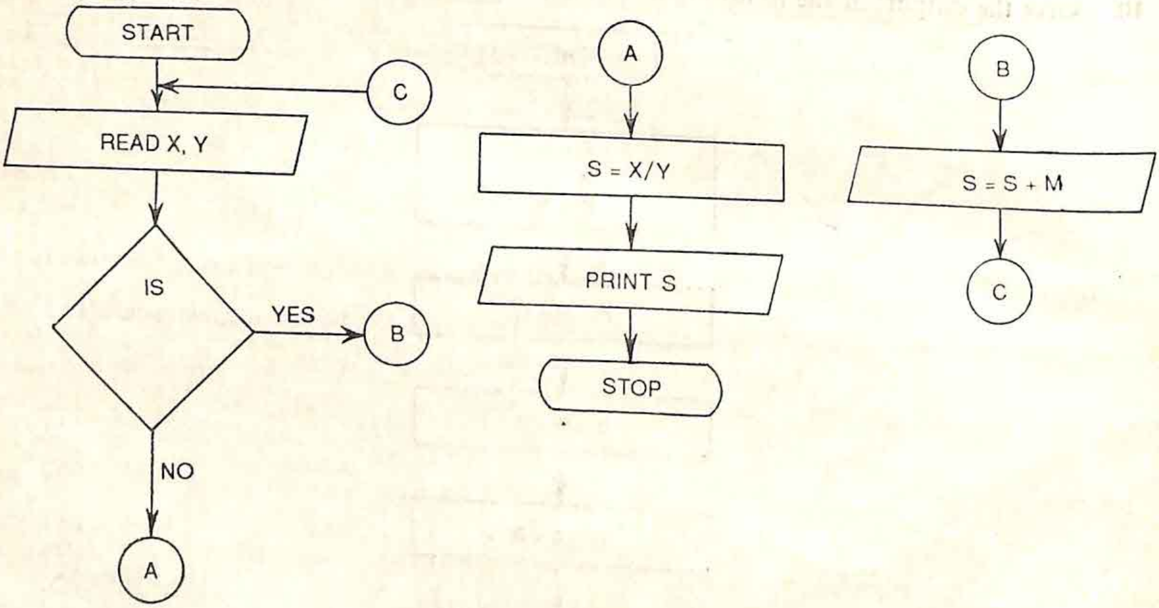


11. Fill in the blanks in the following flow charts :

(*i*) A flow chart for calculating and printing the sum of first 100 numbers.



(*ii*) A flow chart to find the division of X by Y is given below. Give proper check whether Y is equal to zero or not.



Extend the above flow chart to find the quotient and remainder of X/Y.

12. Draw a flow chart for finding the greatest number that will divide 2930 and 3250 and will leave as remainders 7 and 11 respectively.

13. Draw a flow chart for finding the L.C.M. of 364, 2520 and 5265.

14. A house is sold for Rs 1230 at a loss of 18%. What would have been the loss or gain per cent had it been sold for Rs 1600 ? Write the algorithm and draw the flowchart for solving the above problem.

15. A man purchases a certain number of toffees at 25 a rupee and the same number at 20 a rupee. He mixes them together and sells them at 45 for 2 rupees. Find his gain or loss per cent on the entire transaction. Write the step-by-step procedure for solving the above problem.

16. What sum of money will produce Rs 143 interest in $3\frac{1}{4}$ years at $2\frac{1}{2}$ p.c. simple interest ? Draw a flowchart for the above problem.

17. Find the amount at compound interest on Rs. 2700 in 3 years at $3\frac{1}{3}$ per cent p.a. Write the algorithm for solving the above problem.

18. If 30 men working 7 hours a day can do a piece of work in 18 days, in how many days will 21 men working 8 hours a day do the same piece of work ? Write the algorithm and draw the flow chart for above problem.

19. How many horses would be required to plough 117 hectares of land in 35 days, if 10 horses can plough 13 hectares in 7 days ? Draw the flow chart for the above problem.

20. In finding the H.C.F. of two numbers the last divisor is 49 and the quotients 17, 3, 2. Find the numbers. Draw a flow chart for the solution to the above problem.

21. Draw the flow chart for finding the smallest prime number greater than 47.

22. Draw a flow chart for finding the cost price of an item given its selling price and percentage gain or loss incurred.

23. At what rate per cent simple interest will a sum of money treble itself in 25 years ? Write the algorithm for the above problem.

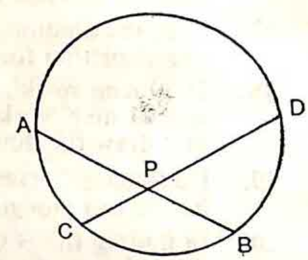
□ □

TEST PAPERS

TEST PAPER 1

(*Time Allowed : 3 hours*)

1. Fill in the blanks to make each of the following a true statement :
   (*a*) Value of $\sin 20° \sin 70° - \cos 20° \cos 70°$ is.........
   (*b*) In the figure, chords AB and CD intersect inside the circle at P. If AP=6 cm, BP=4 cm, PD=8 cm, then CP=......



   (*c*) If P, Q, R are three points on a circle with centre O and $\angle PQR = 57°$, then $\angle POR =$.........
   (*d*) The mean of 3, 5, 7, 9 and 11 is.........
   (*e*) A real number a is said to be a zero of a polynomial $P(x)$, if.........

2. (*a*) Find the sum and the product of the roots of the equation $x^2 - \sqrt{3} = 0$.
   (*b*) Is $\sqrt{100} + \sqrt{36}$ the same as $\sqrt{100+36}$? Give reasons.
   (*c*) What is a system of linear equations having no solution called ?
   (*d*) Find the g.c.d. of $(2x-7)(3x+4)$ and $(2x-7)^2(x+3)$.
   (*e*) Factorise $x^2 - x - 12$.

3. (*a*) Prove that $\dfrac{1}{1+\sin A} + \dfrac{1}{1-\sin A} = \dfrac{2}{\cos^2 A}$
   (*b*) In a circle of 6 cm radius, find the length of a chord that has a central angle of 60°.
   (*c*) Two men are on diametrically opposite sides of a tower. They measure the angles of elevation of the top of the tower as 20° and 24° respectively. If the height of the tower is 40 m, find the distance between them.

4. (*a*) Prove that angles in the same segment of a circle are equal.
   (*b*) AB and CD are two equal intersecting chords of a circle whose centre is O. If M and N are respectively the mid-points of AB and CD, prove that $\angle OMN = \angle ONM$.
   (*c*) In cyclic trapezium ABCD in which AD $\parallel$ BC, show that AB=CD.

5. (*a*) If the corresponding sides of two triangles are proportional, then prove that they are similar.
   (*b*) In $\triangle ABC$, $\angle BAC = 90°$ and segment AD is perpendicular to the hypotenuse BC. Prove that $AD^2 = BD \times DC$.
   (*c*) PQ and PR are equal chords of a circle. Prove that the tangent to the circle at P is parallel to the chord QR.

6. (*a*) Find the real values satisfying $x^4 - 2x^2 - 3 = 0$.
   (*b*) Express $\left(\dfrac{x+1}{x-1} + \dfrac{x-1}{x+1}\right)^2$ as a rational expression.
   (*c*) Some students planned a picnic. The budget for food was Rs. 24. Because four of the group failed to go, the cost of food to each member got increased by Re. 1. How many students attended the picnic ?

7. (*a*) Find the area of a triangle whose sides are 2·22 m, 2·46 m and 1·9 m.

(*b*) A metallic sphere of radius 10·5 cm is melted and then recast into small cones, each of radius 3·5 cm and height 3 cm. Find how many such cones are formed ?

(*c*) A 20-metre deep well with diameter 14 metres is dug up and the earth from digging is spread evenly to form a platform 22 m×14 m. Determine the height of the platform.

8. (*a*) The mean monthly salary paid to 75 employees in a company is Rs. 1420. The mean salary of 25 of them is Rs. 1350 and that of 30 others is Rs. 1425. Find the mean salary of the remaining employees.

(*b*) In a study on certain disease, the following data was obtained. Find the average at first detection.

Age at first detection (*in years*)	*Number of patients*
2—6	1
6—10	9
10—14	21
14—18	47
18—22	52
22—26	36
26—30	19
30—34	3

9. Draw a flow chart to find the sum of cubes of First 10 numbers

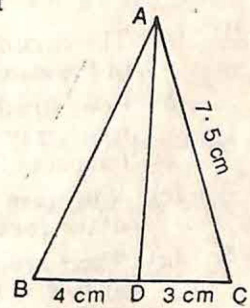
i.e., $1^3+2^3+3^3+\cdots\cdots 10^3$

TEST PAPER 2

(*Time Allowed* : 3 *hours*)

1. Fill in the blanks to make each of the following a true statement :

(*a*) If a perpendicular is drawn from the centre of a circle to a chord, then the foot of the perpendicular·········the chord.



(*b*) In the figure, if AD is the bisector of ∠BAC, then AB=······cm.

(*c*) The value of $\dfrac{\sin 40°}{\cos 50°}$ is·········

(*d*) The zeros of polynomial x^2-9 are·········

(*e*) A system of linear equations which has at least one solution is called a system of·········equations.

2. (*a*) Find the sum and the product of the roots of the equation

$$4x^2-4x-3=0.$$

(*b*) Find the value of k for which the system

$kx+2y=5$, $\quad 3x+y=1$ has unique solution.

(*c*) Find the l.c.m. of $(2x-7)(3x+4)$ and $(2x-7)^2\ (x+3)$.

(*d*) How many transverse common tangents can be drawn to two circles of radii 2·5 cm and 3 cm, their centres being 5 cm apart.

(*e*) Find the measures of two opposite angles of a cyclic quadrilateral, if one of them is $\frac{2}{7}$th of the other.

3. (*a*) Prove that $\dfrac{\sin(90°-A)\cos(90°-A)}{\tan A}=1-\sin^2 A.$

(*b*) Prove that $\dfrac{\sin\theta}{1+\cos\theta}+\dfrac{\sin\theta}{1-\cos\theta}=\dfrac{2}{\sin\theta}$

(*c*) From the top of a cliff 100 metres high, the angles of depression of the top and bottom of a tower are observed to be $(32{\cdot}6)^\circ$ and 45° respectively. Find the height of the tower.

4. (*a*) Prove that a tangent to a circle at any point of it is perpendicular to the radius through the points of contact.

(*b*) Two circles intersect each other at the points P and Q. If AB and AC are tangents to two circles from a point A on the line containing P and Q, prove that AB=AC.

(*c*) AB and CD are equal chords of a circle whose centre is O. When produced, these chords meet at E. Prove that EA=EC.

5. (*a*) Solve $6\left(x^2+\dfrac{1}{x^2}\right)-35\left(x+\dfrac{1}{x}\right)+62=0$

(*b*) Father is six times as old as his son. Four years hence he will be four times as old as his son. Find their present ages.

(*c*) Solve the system of equations

$$11x+15y+23=0$$
$$7x-2y-20=0.$$

6. (*a*) If a line divides any two sides of a triangle in the same ratio, prove that the line is parallel to the third side.

(*b*) If a perpendicular is drawn from the vertex containing the right angle of a right triangle to the hypotenuse, then prove that the triangles on each side of the perpendicular are similar to each other.

(*c*) A point O in the interior of a rectangle ABCD is joined with each of the vertices A, B, C and D. Prove that

$$OB^2+OD^2=OA^2+OC^2.$$

7. (*a*) The circumference of the base of a 9 m high conical tent is 44 m. Find the volume of the air contained in it. (Use $\pi=22/7$)

(*b*) Four equal circles are described about the four corners of a square so that each touches two of the others. Find the area of the space enclosed between the circumferences of the circles, each side of the square measuring 14 cm. (Use $\pi=22/7$)

(*c*) A toy is in the form of a cone mounted on a hemisphere. The diameter of the base of the cone is 6 cm and its height is 4 cm. Compute the surface area of the toy.

8. (*a*) There are 45 students in a class, of which 15 are girls. The average weight of 15 girls is 45 kg and that of the 30 boys is 52 kg. Find the mean weight in kg of the entire class.

(*b*) In a city the following weekly observations were made in a survey of cost of living index for 1970-71. Calculate the average mean weekly cost of living index :

Cost of living index	*Number of weeks*
140–150	5
150–160	10
160–170	20
170–180	9
180–190	6
190–200	2

9. Draw a flow chart to find the H.C.F. of 30, 45 and 75.

TEST PAPER 3

(*Time Allowed : 3 hours*)

1. Fill in the blanks to make the following statements true :

(*a*) Tangent at any point of a circle is.........to the radius through that point.

(*b*) Two circles are congruent if and only if they have equal.........

(*c*) $\sin 50° + \cos 40° = 2 \sin (.........)°$.

(*d*) The solution set of the system of equations

$3x - 4y = -7, \quad 3x - 4y = -9$ is.........

(*e*) The volume of a right circular cone=.........

2. (*a*) Find the zeros of the polynomial x^2+1, $x \in R$.

(*b*) Write down a quadratic equation whose roots are -2 and 3.

(*c*) Find the reciprocal of the rational expression $\dfrac{x^7 - \sqrt{2x}}{8x^7 - \sqrt{3x}}$.

(*d*) Find the length of a chord which is at a distance of 3 cm from the centre of a circle whose radius is 5 cm.

(*e*) Given a point P in the exterior of a circle. How many secants can be drawn through P to the circle ?

3. (*a*) Find the roots of quadratic equation

$$\left(\frac{x}{x-1}\right)^2 + 4\left(\frac{x}{x-1}\right) + 2 = 0, \; x \neq 1$$

(*b*) The sum of the squares of the three consecutive natural numbers is 110. Determine the numbers.

(*c*) Simplify : $\dfrac{x-1}{x-2} - \dfrac{x+1}{x+2} - \dfrac{4}{4-x^2} + \dfrac{2}{2-x}$.

4. (*a*) Prove that the angle subtended by an arc at the centre is double the angle subtended by it any point on the remaining part of the circle.

(*b*) Prove that a cyclic parallelogram is a rectangle.

(*c*) Prove that the tangents at the end-points of a diameter of a circle are parallel.

5. (*a*) Prove that if a line is drawn parallel to one side of a triangle, the other two sides are divided proportionally.

(*b*) ABCD is a trapezium such that AB ∥ DC. If O is the point of intersection of its diagonals, then show that $\dfrac{OA}{OC} = \dfrac{OB}{OD}$.

(*c*) In △ABC, AD is perpendicular to BC.

Prove that $\quad AB^2 - BD^2 = AC^2 - CD^2$.

6. (*a*) A sector is cut from a circle of radius 21 cm. The angle of the sector is 150°. Find its length and area.

(*b*) A heap of wheat is in the form of a cone of diameter 9 m and height 3·5 m. Find its volume. How much canvas cloth is required to just cover the heap ?

(Use $\pi = 3{\cdot}14$)

(*c*) The volumes of a sphere and a right cylinder are equal and the diameter of the sphere equals the diameter of the base of the cylinder. Determine the ratio of the height of the cylinder to the diameter of the sphere.

7. (*a*) Find the value of $\dfrac{\cos 59°}{\sin 31°}$.

(*b*) Prove that $\frac{1+\sin A}{\cos A}+\frac{\cos A}{1+\sin A}=\frac{2}{\cos A}$.

(*c*) A kite flying at a height of 65 metres is attached to a string inclined at 31° to the horizontal. What is the length of the string? Assume that the string is tight.

8. (*a*) Following is the distribution of earnings of 200 workers in a flour mill :

Monthly wages (in rupees)	*No. of workers*
80—100	20
100—120	30
120—140	20
140—160	40
160—180	90

Find the average earnings of the workers.

(*b*) The mean of 30 values was 150. It was detected on rechecking that one value 165 was wrongly copied as 135 for the computation of the mean. Find the correct mean.

9. A bicycle is sold at a profit of 12%. Had it been sold for Rs. 180 more, 18% would have been gained. Find its cost price. Draw the flow chart for solving the above problem.

TEST PAPER 4

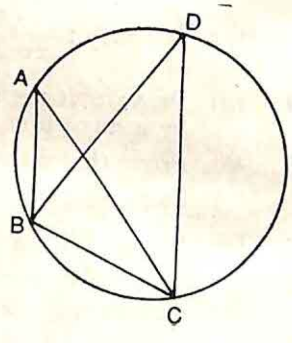
(*Time Allowed* : *3 hours*)

1. Fill in the blanks, making each of the following a true statement :

(*a*) Equal chords of a circle are.........from the centre.

(*b*) The value of $\frac{\cos 20° 35'}{\sin 69° 25'}$ is.........

(*c*) In the figure, if $\angle BDC=30°$ and $\angle CBA=110°$, then $\angle BCA=$.........



(*d*) Two tangents to a circle from an external point are.........

(*e*) $3x^3+7x^2+5$ is a polynomial of degree......... in x.

2. (*a*) Find the value of k so that the equation $9x^2+3kx+4=0$ has equal roots.

(*b*) Reduce the rational expression $\frac{x+1}{2x^2+x-1}$ to its lowest terms.

(*c*) The ratio of any two corresponding sides of two similar triangles is 3 : 2. Find the ratio of the areas of these triangles.

(*d*) When do two linear equations in x and y have no common solution ?

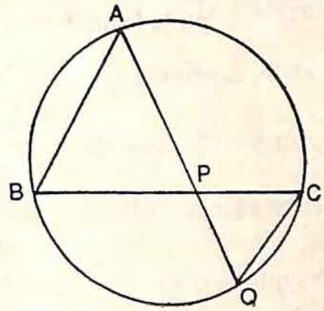
(*e*) What is the total surface area of a hemisphere whose radius is x cm ?

3. (*a*) Solve $\frac{1}{x+7}+\frac{1}{x+3}=\frac{6}{5}$, $x \in R$.

(*b*) Solve the system of equations :

$$5x+2y+13=0 \quad \text{and} \quad 7x-5y+26=0.$$

(*c*) The length of a room is 3 metres more than its breadth. If the area of the room is 70 sq. metres, determine the dimensions of the room.



4. (*a*) Prove that if two circles touch each other, the point of contact lies on the line joining their centres.

(*b*) ABCD is a cyclic quadrilateral. A circle passing through A and B meets AD and BC in the points E and F respectively. Prove that EF is parallel to DC.

(*c*) In the figure, P is a point on the chord BC such that AB=AP. Prove that CP=CQ.

5. (*a*) Prove that $\frac{\cos A}{1-\tan A}+\frac{\sin^2 A}{\sin A-\cos A}=\sin A+\cos A$.

(*b*) Show that $\cos\theta \cos(90°-\theta)-\sin\theta \sin(90°-\theta)=0$.

(*c*) A ladder leaning against a vertical wall makes an angle of 20° with the ground. The foot of the ladder is 3 metres from the wall. Determine the length of the ladder.

6. (*a*) Prove that in a right triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

(*b*) If two sides and a median bisecting one of these sides of a triangle are respectively proportional to the two sides and the corresponding median of another triangle, then prove that the two triangles are similar.

(*c*) Prove that the bisector of an angle of a triangle divides the opposite side in the ratio of the sides containing the angle.

7. (*a*) The length of the minutes hand of a clock is 14 cm. Find the area swept by the minutes hand in one minute and in one hour. (Take $\pi=3{\cdot}14$)

(*b*) A cone of height 24 cm, has a curved surface area 550 cm². Find its volume. $\left(\text{Take } \pi=\frac{22}{7}\right)$

(*c*) A cylindrical jar of radius 6 cm contains oil. Iron spheres, each of radius 1·5 cm are immersed in the oil. How many such spheres are necessary to raise the level of the oil by 2 cm ?

8. (*a*) The mean of the following frequency table is 50. But the frequencies f_1 and f_2 in classes 20—40 and 60—80 are missing. Find the missing frequencies :

Class	*Frequency*
0—20	17
20—40	f_1
40—60	32
60—80	f_2
80—100	19
Total	120

(*b*) In a study to test a new variety of wheat, an experiment was performed on 50 similar plots (under similar conditions) and the following results were obtained :

Yield per hectare (*in quintals*)	*Number of fields*
10–20	2
20–30	7
30–40	12
40–50	15
50–60	8
60–70	6

Find the mean yield per hectare from the above data.

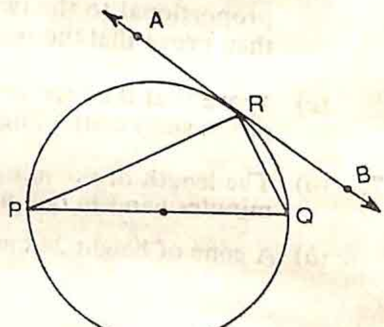
9. A man had Rs. 2,000, part of which he lend at 5% and the rest at 4%. The whole annual interest received was Rs. 92. How much did he lent at 5% ? Write the algorithm for the above problem and draw its flow chart.

TEST PAPER 5

(*Time Allowed* : *3 hours*)

1. Fill in the blanks to make the following statements true :

(*a*) In the figure, if AB is a tangent to the circle at R, PQ is a diameter of the circle and $\angle RPQ = 25°$, then $\angle ARP =$........



(*b*) If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is........

(*c*) The value of $\dfrac{\sin 10°}{\cos 80°}$ is........

(*d*) The solution of the system of equation $x+y=5$, $x-y=3$ is........

(*e*) If the discriminant D of a quadratic equation $ax^2+bx+c=0$, $a \neq 0$, is zero, the roots are........

2. (*a*) One root of the equation $3x^2-10x+3=0$ is $\frac{1}{3}$. Find the other root.

(*b*) Express $(x^2+2)+\dfrac{2x}{x+1}$ as a rational expression.

(*c*) Find the measures of two opposite angles of a cyclic quadrilateral, if one of them is $\dfrac{11}{4}$ th of the other.

(*d*) The perimeters of two similar triangles are 24 cm and 18 cm respectively. If one side of the first triangle is 8 cm, find the corresponding side of the other triangle.

(*e*) The mean of the numbers 6, y, 7, x, 14 is 8. Express y in terms of x.

3. (*a*) Prove that $\left(\dfrac{1+\cos\theta}{\sin\theta}\right)^2=\dfrac{1+\cos\theta}{1-\cos\theta}$.

(*b*) Prove that $\sin\theta\cos(90°-\theta)+\cos\theta\sin(90°-\theta)=1$.

(*c*) The shadow of Qutab Minar is 81 metres long when the angle of elevation of the sun is 41° 30′. Find the height of Qutab Minar.

4. (*a*) Prove that the perpendicular from the centre of a circle to a chord bisects the chord.

(*b*) If the sides of a quadrilateral touch a circle, prove that the sum of a pair of opposite sides is equal to the sum of the other pair.

(*c*) P is the mid-point of an arc APB of a circle. Prove that the tangent at P is parallel to the chord AB.

5, (*a*) Prove that the ratio of the areas of two similar triangles is equal to the ratio of the squares of any two corresponding sides.

(*b*) In $\triangle ABC$, $\angle BCA$ is a right angle. Q is the mid-point of the side BC. Prove that $BC^2=4\ (AQ^2-AC^2)$.

(*c*) Prove that any line parallel to parallel sides of a trapezium divides the non-parallel sides proportionally.

6. (*a*) Following are the data on the daily wages of casual labour employed by a group of limited concern :

Daily wages (in rupees)	*Frequency*
4– 6	6
6– 8	5
8–10	10
10–12	8
12–14	3
14–16	2
Total	34

(*b*) A school has 4 sections of chemistry in class XI having 40, 35, 45 and 42 students. The mean marks obtained in chemistry test are 50, 60, 55 and 45 respectively for the 4 sections. Determine the overall average of marks per student.

7. (*a*) Solve $\sqrt{x^2-16}-(x-4)=\sqrt{x^2-5x+4}$.

(*b*) Find the quadratic equation whose roots are reciprocals of the roots of the equation

$$3x^2-20x+17=0.$$

(*c*) Solve the following system of equations by graphical method :

$$2x-y=9, \qquad 5x+2y=27.$$

8. (*a*) How many spherical bullets can be made out of a cube of lead whose edge measures 22 cm, each bullet being 2 cm in diameter ? $\left[\text{Take } \pi=\frac{22}{7}\right]$

(*b*) Given a circle with radius 3·5 cm. Find the area of its sector with central angle 30°.

(*c*) A well with 10 metres inside diameter is dug 14 metres deep. Earth taken out of it has been spread evenly around it to a width of 5 cm. Find the height of the embarkment so formed.

9. A fort had provisions for 150 men for 45 days. After 10 days 25 men left the fort. How long will the food last at the same rate ? Draw a flow chart for solving the above problem.

Exercise 1 (g)

1. (*a*) Inconsistent (*b*) Consistent (*c*) Dependent (*d*) Dependent
2. (*a*) A unique solution (*b*) Infinitely many solutions (*c*) No solutions (*d*) A unique solution
3. (*a*) $k \neq \frac{-10}{3}$ (*b*) $k = \frac{-10}{3}$
4. (*a*) $k \neq 6$ (*b*) $k=6$

Exercise 1 (h)

1. $x=2, y=3$
2. $x=0, y=-3$
3. $x=-5, y=-7$
4. $x=6, y=-\frac{1}{2}$
5. $x=6, y=2$
6. $x=12, y=9$
7. $x=a, y=b$
8. $x=b, y=a$
9. $x=\frac{p^2+q}{1-pq}, y=\frac{q^2+p}{1-pq}$
10. $x=\frac{pr}{p^2+q^2}, y=\frac{qr}{p^2+q^2}$
11. $x=\frac{2a+b}{2}, y=\frac{2a-b}{2}$
12. $x=\frac{a^2+b^2}{2ab}, y=\frac{b^2+2ab-a^2}{2ab}$

Exercise 1 (i)

1. 30, 15
2. 11, 4
3. 5, 11
4. 12, 15
5. 35, 71
6. 12
7. 36
8. 16
9. 24
10. 49
11. 69
12. 84
13. 6
14. $\frac{5}{7}$
15. $\frac{11}{18}$
16. $\frac{2}{3}$
17. $\frac{5}{12}$
18. $\frac{3}{5}$
19. $\frac{3}{7}$
20. 42 yrs., 18 yrs.
21. Father 40 yrs., Son 10 yrs.
22. Father 34 yrs., Son 12 yrs.
23. 20 years, 24 years
24. 33 yrs., 12 yrs.
25. Rs. 4, Rs. 9
26. 4 g, 10 g
27. 7, 43
28. 34, 80
29. 5
30. Adults 1756, Children 744
31. Rs. 10

Review Exercise I

1. (*a*) linear (*b*) line (*c*) y (*d*) x (*e*) no (*f*) consistent
2. -3
3. $x=3, y=0$
4. $x=1, y=-2$
5. $x=5, y=-4$
6. $x=1, y=4$
7. $x=20, y=12$
8. $x=6, y=1$
9. (1·5, 1)
10. $x=-\frac{1}{2}, y=\frac{1}{2}$
11. 10 paise 45, 25 paise 15
12. Rs. 35
13. 27
14. $\frac{3}{4}$
15. 19
16. $x=\frac{1}{14}, y=\frac{1}{6}$
17. Father 33 yrs., Son 10 yrs.
18. A, 100; B, 80
19. 34, 62
20. Sailor 10 km/hour, current 2 km/hour.

QUADRATIC EQUATIONS

Exercise 2 (a)

1. (*a*), (*b*), (*c*), (*d*) and (*g*) are quadratic equations
4. -4,
5. $\frac{2}{3}, \frac{3}{2}$
6. $-2, \frac{1}{2}$
7. $\frac{-3}{2}, \frac{5}{3}$.

ANSWERS

LINEAR EQUATIONS IN TWO VARIABLES

Exercise 1 (b)

1. (4, 1) **2.** (2, −1) **3.** (2, 8) **4.** (1, 2)

5. (9, 2) **6.** (0, −2) **7.** (1, 1·5) **8.** $\left(\frac{5}{7}, \frac{22}{7}\right)$

9. (−1, 1) **10.** (−1, 3)

11. (4, 3) ; Consistent **12.** Infinite solutions ; Dependent

13. No solution ; Inconsistent **14.** Consistent ; (−2, 1)

15. (2, 3) ; Consistent **16.** (8, 1) ; Consistent.

Exercise 1 (c)

1. $x=7, \quad y=3$ **2.** $x=1, \quad y=4$ **3.** $x=3, \quad y=1$

4. $x=2{\cdot}5, \quad y=2$ **5.** $x=3, \quad y=-1$ **6.** $x=1, \quad y=-3$

7. $x=\frac{1}{7}, \quad y=\frac{22}{7}$ **8.** $x=\frac{6}{5}, y=-\frac{2}{5}$ **9.** $x=\frac{11}{3}, y=\frac{19}{3}$

10. $x=\frac{-25}{13}, y=\frac{-97}{13}$

Exercise 1 (d)

1. $x=2, \quad y=-1$ **2.** $x=2{\cdot}5, y=1$ **3.** $x=3, \quad y=2$

4. $x=2, \quad y=1$ **5.** $x=2, \quad y=5$ **6.** $x=-4, y=2$

7. $x=\frac{4}{5}, y=\frac{7}{5}$ **8.** $x=\frac{5}{2}, y=3$ **9.** $x=3, \quad y=\frac{11}{7}$

10. $x=6, \quad y=\frac{-1}{2}$

Exercise 1 (e)

1. $x=-1, \ y=-1$ **2.** $x=1, \quad y=1$ **3.** $x=5, \quad y=3$

4. $x=2, \quad y=1$ **5.** $x=1, \quad y=-\frac{1}{2}$ **6.** $x=\frac{-5}{4}, y=-\frac{1}{4}$

7. $x=5, \quad y=\frac{1}{5}$ **8.** $x=2, \ y=-3$ **9.** $x=\frac{7}{2}, \quad y=0$

10. $x=-1{\cdot}1, y=-{\cdot}8$

Exercise 1 (f)

1. $x=\frac{21}{11}, y=\frac{8}{11}$ **2.** $x=1{\cdot}1, y=0{\cdot}8$ **3.** $a=3, \quad b=\frac{1}{2}$

4. $x=3, \quad y=-2$ **5.** $x=10, \ y=4$ **6.** $x=3, \ y=2$

7. $x=\frac{1}{2}, y=-1$ **8.** $x=\frac{1}{3}, y=-1$ **9.** $x=\frac{1}{14}, y=\frac{1}{6}$

10. $x=3{\cdot}2, y=2{\cdot}3$ **11.** $x=10, y=11$ **12.** $x=3, y=2$

13. $x=\frac{1}{4}, y=\frac{1}{3}$ **14.** $x=-\frac{1}{2}, y=\frac{1}{4}$.

Exercise 2 (b)

1. $5, -5$ 2. $\frac{3}{4}, \frac{-3}{4}$ 3. $\frac{3}{2}, \frac{-3}{2}$
4. $4, -4$ 5. $6, -2$ 6. $9, -3$
7. $0, 3$ 8. $0, \frac{9}{5}$ 9. $0, \frac{3}{2}$
10. $0, \frac{3a}{2}$ 11. $0, 5$ 12. $0, \frac{-9}{4}$
13. $3, 7$ 14. $8, -8$ 15. $\frac{3\sqrt{2}}{2}, \frac{-3\sqrt{2}}{2}$
16. $\frac{1}{\sqrt{11}}, -\frac{1}{\sqrt{11}}$ 17. $5, -5.$

Exercise 2 (c)

1. $2, -8$ 2. $3, 4$ 3. $-3, -5$ 4. $5, -1$
5. $\frac{5}{2}, \frac{-7}{2}$ 6. $-2, \frac{-5}{3}$ 7. $\frac{3}{2}, \frac{-5}{3}$ 8. $\frac{3}{4}, \frac{-2}{3}$
9. $-5, \frac{-2}{3}$ 10. $4, \frac{-2}{3}$ 11. $-2, \frac{-3}{5}$ 12. $\frac{1}{2}, \frac{5}{2}$

Exercise 2 (d)

1. $1+\sqrt{2}, 1-\sqrt{2}$ 2. $\frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$ 3. $\frac{-1+\sqrt{29}}{2}, \frac{-1-\sqrt{29}}{2}$
4. $3+\sqrt{74}, 3-\sqrt{74}$ 5. $\frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$ 6. $\frac{2+2\sqrt{46}}{3}, \frac{2-2\sqrt{46}}{3}$
7. $0{\cdot}78, 3{\cdot}22$ 8. $-1{\cdot}27, 2{\cdot}77$ 9. $2{\cdot}18, 0{\cdot}15$
10. $0{\cdot}65, 7{\cdot}65$ 11. $-0{\cdot}17, 2{\cdot}92$ 12. $-0{\cdot}59, 2{\cdot}26$

Exercise 2 (e)

1. (*a*) 16 (*b*) 1 (*c*) -5 (*d*) 32
2. (*a*) Real roots (*b*) Real roots (*c*) No real roots (*d*) Real roots
3. (*a*) No real roots (*b*) Real roots (*c*) No real roots (*d*) Real roots
4. (*a*) $k \leqslant 4$ (*b*) $k \leqslant 0$ (*c*) $k \geqslant \pm 6$ (*d*) $k \geqslant 2\sqrt{6}$ or $k \geqslant -2\sqrt{6}$
5. (*a*) $\frac{1}{\sqrt{2}}$ (*b*) $\frac{-2}{\sqrt{3}}$ 6. (*a*) No real roots (*b*) $\frac{1}{2}, -1$
(*c*) No real roots (*d*) $\frac{2}{3}, \frac{3}{2}$ 7. (*a*) $4, \frac{5}{2}$ (*b*) $\frac{13\sqrt{7}}{7}, -\sqrt{7}$
8. (*a*) ± 8 (*b*) $\frac{-4}{3}$ 10. $k=2, \frac{-10}{9}$

Exercise 2 (f)

1. $2; 1$ 2. $-3; -5$ 3. $\frac{-5}{4}; -\frac{1}{2}$ 4. $\frac{4}{3}; \frac{1}{3}$
5. $-p; q$ 6. $\frac{q}{p}; \frac{r}{p}$ 7. $x^2-2x+2=0$ 8. $x^2-4x+5=0$
9. $x^2+3x+4=0$ 10. $6x^2+x-2=0$ 11. $x^2-3x-10=0$ 12. $4x^2+8x+3=0$
13. $x^2-10x+22=0$ 14. $9x^2-12x-1=0$ 15. $x^2-12=0$ 16. $1; k=-6$
17. $k=-3; -\frac{1}{2}$ 18. $k=4$

Exercise 2 (g)

1. (*a*) 6 (*b*) $\frac{2}{3}$ (*c*) -2

2. (*a*) $\frac{p^2-q}{q}$ (*b*) $(p^2-2q)\,q$ (*c*) p^3-3pq

3. (*a*) $\frac{a-b+c}{a}$ (*b*) $\frac{b^4-4ab^2c}{a^2c^2}$ (*c*) $\frac{3abc-b^3}{c^3}$

Exercise 2 (h)

1. (*a*) Real factors (*b*) No real factors (*c*) No real factors (*d*) Real factors

2. (*a*) $k\leqslant 4{\cdot}5$ (*b*) $k\leqslant 3\frac{1}{8}$ (*c*) $k\geqslant\frac{-2}{3}$ (*d*) $k\geqslant-\frac{9}{16}$

3. (*a*) $(x-1)\,(2x-1)$ (*b*) $(x+5)\,(2x+1)$
   (*c*) $(2x+3)\,(3x-7)$ (*d*) $\frac{1}{2}\,(x-2)\,(x-4)$

4. (*a*) $(x+2-\sqrt{2})\,(x+2+\sqrt{2})$ (*b*) $2\left(x+\frac{3+\sqrt{65}}{4}\right)\left(x+\frac{3-\sqrt{65}}{4}\right)$
   (*c*) $(x+5-3\sqrt{3})\,(x+5+3\sqrt{3})$ (*d*) $3\left(x+\frac{3+\sqrt{15}}{3}\right)\left(x+\frac{3-\sqrt{15}}{3}\right)$

Exercise 2 (i)

1. $x=\pm2,\ x=\pm3$
2. $x=\pm\sqrt{\frac{2}{5}}$
3. $x=\pm1,\ x=\pm\sqrt{\frac{3}{2}}$
4. $x=\pm4,\ x=\pm\frac{2}{3}$
5. $x=-1,\ x=1{\cdot}5$
6. $x=\frac{5}{12},\ x=\frac{1}{4}$
7. $x=1,\ x=3$
8. $x=-2,\ x=-3$
9. $x=1,\ x=\frac{4}{9}$
10. $x=\frac{1}{4}$
11. $x=2,\ x=1\frac{1}{2}$
12. $x=27,\ x=64$
13. $x=6,\ x=11$
14. $x=2,\ x=5$
15. $x=0,\ x=2$
16. $x=2,\ x=5.$

Exercise 2 (j)

1. $x=5$
2. $x=0,$
3. $x=0$
4. $x=3,\ -1$
5. $x=-1$
6. $x=3,\ x=\frac{1}{3},$
7. $x=1$
8. $x=1,\ x=-1$
9. $x=-1$
10. $x=-4,\ \frac{-4}{7}$
11. $x=2,\ \frac{-1}{2},\ -1\pm\sqrt{2}$
12. $x=1,\ \frac{1}{5}$
13. $x=-1,\ 1,\ \frac{3+\sqrt{13}}{2},\ \frac{3-\sqrt{13}}{2}$
14. $x=\frac{4}{13},\ \frac{9}{13}$
15. No solution
16. $x=\frac{3}{2},\ -2.$

Exercise 2 (k)

1. 9 2. 19, 21 *or* −21, −19 3. 6, 8 4. 12, 13
5. 16, 17 6. 7, 8, 9 7. 12, 15 *or* −12, −15
8. 23 9. 6, 9 10. 3 11. 4·5 metres/minute
12. 9, 32 13. 7, 8 14. $\frac{3}{5}$, $\frac{5}{3}$
15. 8 cm, 15 cm, 17 cm 16. $x=3$ 17. 6 cm, 4 cm
18. 16 m 19. $\frac{-5+\sqrt{533}}{2}$ cm
20. 5 cm, 12 cm 21. 6 m, 3 m 22. 6 hours 40 minutes
23. $3(\sqrt{5}-1)$ cm internally; $-3(\sqrt{5}+1)$ cm externally
24. 9, 10, 11 25. Father 42 years, Son 3 years
26. 6 years
27. (*i*) $\frac{200}{x}$ hours, $\frac{200}{x+5}$ hours (*ii*) 20 km
28. $x=100$ 29. 8.
30. Base=15 cm, Altitude=8 cm, Hypotenuse=17 cm.

Review Exercise II

1. (*a*) −92 (*b*) $p^2-4q<0$ (*c*) $q^2-4pr=0$ (*d*) $x^2-2x-15=0$
   (*e*) −5 (*f*) $x^2-2x-2=0$ (*g*) two (*h*) $\frac{3}{2}$
2. (*a*) $-\frac{7}{3}$ (*b*) 1 3. $x^2-6x+7=0$
4. (*a*) Two real distinct roots (*b*) Real repeating root
5. $x=\frac{9}{5}$, $x=-\frac{4}{3}$ 6. $y=\frac{1}{3}$, $y=-\frac{1}{16}$
7. $y=1$, $y=\frac{1}{10}$ 8. $y=4$, $y=\frac{13}{2}$
9. $x=4$, $x=-1$ 10. 5, 6, 7
11. Length 10 m, Breadth 7 m
12. $(3-\sqrt{5})$ m *i.e.* 76 cm nearly
13. 3 km/hour 14. 20 m × 5 m
15. 7 km/hour

RATIONAL EXPRESSIONS

Exercise 3 (a)

1. $(x-3)(x+1)^2$ 2. $2(x-1)(x+1)^2$
3. $x-2$ 4. $2x+3$
5. $4x^2(2x+1)$ 6. $(x+3)(x-6)(x-2)^2$
7. $(x+3)(x+4)^2$ 8. $-(x+1)(x+3)(x-2)$
9. $-(x+3)^2(2x+1)(3x-4)$ 10. $(x^2+2x-3)(x^3-x^2-5x+2)$

Exercise 3 (b)

1. (*a*) and (*c*) are rational expressions.
2. $\frac{ax+b}{cx^2+dx+e}$, $a\neq 0$, $c\neq 0$.

3. $\dfrac{ax^m+b}{cx^n+dx^p+ex^r}$, where m, n, p, r are natural numbers and a, b, c, d, e are non-zero real numbers.

4. $\dfrac{x^2+x-2}{3x^2-7x+2}$.

Exercise 3 (c)

1. $\dfrac{(x+3)(x-2)}{x-1}$ 2. $\dfrac{(x+2)(x-3)}{(x+1)(x-2)}$ 3. $\dfrac{3}{5}$ 4. $-\dfrac{1}{x+3}$

5. $\dfrac{x-3}{x+1}$ 6. $\dfrac{x+3}{2x^2}$ 7. $\dfrac{x-4}{x-3}$ 8. $\dfrac{x+1}{2x+1}$

9. $\dfrac{2x+1}{2x-3}$ 10. $\dfrac{5(3x-1)}{x-5}$ 11. $\dfrac{2x-5}{3x-2}$ 12. $\dfrac{2x+y}{2x+3y}$

Exercise 3 (d)

1. $\dfrac{x+a}{x-a}$ 2. $\dfrac{2x-5}{x+5}$ 3. $\dfrac{2x^2-2}{x+3}$ 4. 1

5. $\dfrac{x^2+y^2}{xy}$ 6. $\dfrac{a-c}{ac}$ 7. $\dfrac{2x^2+2x-7}{(x+3)(x-2)}$ 8. $\dfrac{x+6}{9x(x+1)}$

9. $\dfrac{2x-6}{x^2-36}$ 10. $\dfrac{2(x^2+1)}{(x-1)^2(x+1)}$ 11. $\dfrac{2x^3}{x^2-y^2}$ 12. $\dfrac{2(x-1)}{(x+1)(25x^2-1)}$

13. $\dfrac{x^5+x^4+3x^2+x-3}{(x^2-1)(x^3+2)}$ 14. $\dfrac{4x^3}{x^4-y^4}$ 15. $\dfrac{3x^2-14}{(x-1)(x-2)(x-3)}$

Exercise 3 (e)

1. $\dfrac{0}{1}$ 2. $\dfrac{-(x^2+1)}{x-1}$ 3. $\dfrac{-x^2+3x}{x+2}$ 4. $\dfrac{3}{x-y}$

5. $\dfrac{2}{x-y}$ 6. $\dfrac{4x}{x^2-1}$ 7. $\dfrac{-20x}{x^2-25}$ 8. $\dfrac{-x^2+8x-3}{x^2-1}$

9. $\dfrac{-x+8}{(x-2)^2}$ 10. $\dfrac{x^3+6x-1}{x^2-1}$ 11. 1 12. $\dfrac{-3x^3-x^2+2}{x^2-1}$

13. $\dfrac{3}{x-2}$ 14. $\dfrac{2x^5-2x^4-4x^3-14x^2-4x+4}{(x-1)(x-2)(x+2)(2x+1)}$.

Exercise 3 (f)

1. $\dfrac{2(x^2+4x+3)}{x^2-3x+2}$ 2. $\dfrac{x^3+x^2+x+1}{x^3-x^2-2x+2}$ 3. $2(x+2)$

4. x^2-5x+6 5. $\dfrac{x^2-2x-24}{x^2-2x-35}$ 6. x

7. $\dfrac{a+b}{a-x}$ 8. $\dfrac{2x+1}{x+1}$ 9. $\dfrac{1-y}{x(1-x)}$

10. $\dfrac{1}{x-y}$ 11. $\dfrac{x+1}{x-2}$

Exercise 3 (g)

1. $\dfrac{1}{1}$ 2. $\dfrac{qx}{px}$; $p(x)\neq 0$ 3. (a) x^2

(b) $\dfrac{x-1}{x+1}$ (c) $\dfrac{x-1}{x^2+x+1}$ 4. $\dfrac{x^2+x-12}{x^2+x-2}$

4. 28,000 cm^3 5. 14 cm 6. 25,000
7. 1·2 m 8. 2 m 9. 1·57 m^2
11. 1502 dm^3 12. 4300 metric tonnes 13. ·005 cm
14. 9450 l : 111 mm 15. 10,00,000 l 16. 20 cm, 16 cm, 8 cm
17. 12 cm, 8 cm 18. 13·9 cm 19. $1\frac{47}{73}$ dm
20. 1 cm.

Exercise 5 (b)

1. 3375 cm^3 ; 1350 cm^2 2. (*i*) 96 cm^2 (*i*) 64 cm^3
3. (*i*) 150 cm^2 (*ii*) 125 cm^3 4. 27
5. 1440 cm^2 6. 1331 cm^3 7. 18
8. 7·57 dm 9. 60 cm 10. 120
11. 864 cm^2 12. 1440 cm^2 13. 514·3 cm^2
14. 38 cm^3 ; 3·36 cm.

Exercise 5 (c)

1. 2200 m^3 2. 880 m^2 3. 15,092 metric tonnes
4. 1650 cm^3 ; 660 cm^2 5. (*a*) 62·86 m^3 ; Rs. 628·57 6. (*i*) 40 cm (*ii*) 38,500 cm^3
7. (*i*) 15·9 cm (*ii*) 5000 cm^3 8. 6600 dm^3
9. 15 dm 10. 1810·3 dm^3 11. 440 sq. cm
12. 502·66 cm^2 13. 306·2 cm^3 14. 489 nearly
15. 1·17 dm 16. 509 cm 17. $4\frac{2}{3}$ cm
18. 660 l 19. 30 hrs. 41 min. 24 sec.
20. (*a*) 1155 litres (*b*) 24 minutes.

Exercise 5 (d)

1. 550 cm^2 2. 21·6 π cm^2 3. 934·67 cm^3
4. 416 m^3 5. 1584 cm^2 6. 37·7 cm^3
7. 35 m 8. 16 cm 9. $4\frac{2}{3}$ m
10. 1232 cm^3 11. 25,024 sq. m ; 2,58,720 m^3 12. 24 dm ; 25 dm ; 14 dm
13. 10 cm 14. 6 cm
15. Radius=5 m, slant height=13 m 16. 4 cm
17. 754·3 cm^3 18. (*i*) 36 cm, (*ii*) 43·3 cm 19. 603·4 cm^3

Exercise 5 (e)

1. 1386 dm^2 ; 4851 dm^3 2. 616 cm^2 ; 1437·3 cm^3 3. 35 cm
4. 21 cm 5. 718·67 l 6. 28·875 cm^2
7. (*i*) 4 : 1 (*ii*) 16 : 1 8. 2541
9. 10,500 10. 2 cm 11. 3 cm
12. 33 cm 13. 3 cm 14. 1 cm
15. 113·79 dm^3 16. 150 17. 5
18. 5 cm 19. 0·54 cm 20. 100

Review Exercise IV

1. (*a*) False (*b*) False (*c*) True (*d*) False (*e*) False
2. (*a*) 13 cm (*b*) $4\pi r^2$ (*c*) $\frac{4}{3}\pi r^3$ (*d*) four times (*e*) perpendicular
(*f*) $\pi r(l+r)$ (*g*) $\pi R^2 H$ (*h*) three (*i*) $\frac{1}{3}\pi r^2 h$ (*j*) 9 m

5. $\frac{x+2}{x-2}$ 6. $\frac{x-6}{x-7}$ 7. $\frac{(x+y)^2}{y}$

8. $\frac{1}{2x-3}$ 9. $\frac{(x-5)^2}{x^2-10x+9}$ 10. $\frac{x+2}{x^2-5x+6}$

11. $\frac{6x^4+9x^3+15x^2}{x^4-2x^2+1}$ 12. $\frac{x^2+4x+3}{(x+2)^2}$.

Review Exercise III

1. (*a*) integers (*b*) rational (*c*) closed
   (*d*) commutative (*e*) associative (*f*) 1

2. $\frac{x-5}{x^2+2}$ 3. $\frac{x-1}{x^2+1}$ 4. $\frac{x-4}{x+1}$

5. $\frac{2x^3+3x^2+1}{(x+2)(x^2-1)}$ 6. $\frac{x^3-x^2+4}{x^2+2}$ 7. $\frac{x^2-5x+6}{x-4}$

8. $\frac{x^3-7x^2-x+7}{x^3+5x^2+9x+9}$ 9. $\frac{1}{a^8-1}$ 10. $-\frac{x^2+ax+a^2}{x^2}$

11. $\frac{5x^4+46x^3+81x^2+18x+18}{6x^2+48x+72}$.

MENSURATION—PLANE FIGURES

Exercise 4 (a)

1. 63 2. 30 m, 23 m 3. 60%
4. Rs. 4476·36 5. 77·2 m 6. 24 cm or 10 cm
7. 48 cm, 1320 cm² 8. 14 dm, 6 dm 9. Rs. 432.
10. 3 cm, 4 cm; 6 cm².

Exercise 4 (b)

1. 6550 sq. m 2. 7525 m² 3. 3975 m²
4. 27,200 m² 5. 3,01,200 m² 6. 94,150 m²

Exercise 4 (c)

1. 66 cm 2. 154 m² 3. 94·2 g
4. Rs. 3388 5. (*a*) Rs. 1355·20 (*b*) Rs. 1056
6. 1353·625 m² 7. Square by 2·9 cm² 8. 7 m
9. 1·68 cm² 10. 15·84 km/hour 11. 2205 m².

Exercise 4 (d)

1. 22 cm 2. 3·85 cm² 3. 60 cm²
4. 13·2 m² 5. 57·75 cm² 6. 9·46 cm, 4·07 cm²
7. 36° 8. 22 cm² 9. $\frac{5}{12}$
10. $\frac{1}{3}$, 4 : 5 : 3.

Exercise 4 (e)

1. 20·4 cm² 2. 7·125 cm², 71·375 cm²
3. 270·862 m² 4. 285·5 cm², 28·5 cm²

MENSURATION—SOLIDS

Exercise 5 (a)

1. 432 m²; 960 m² 2. 2940 *l* 3. Rs. 230·40

3. 24,000 cm² 4. $112{\cdot}5$ m² 5. 512 cm³
6. 500 7. $4r$ 8. 20 : 27
9. $\sqrt[3]{66}$ cm 10. tiles 960 ; $\frac{7}{20}$ 11. 210 m, 168 m
12. $69{\cdot}71$ cm; $627{\cdot}39$ cm² 13. 216 sq. m 14. $9{\cdot}89$ cm
15. length 14 m, breadth 10 m 16. $12\sqrt{3}$ cm 17. $59{\cdot}4$ metres, Rs. $742{\cdot}50$
18. 42,240 cm³ 19. $37{\cdot}7$ dm³ 20. $2{\cdot}1$ cm
21. $4{\cdot}2$ cm 22. 24,300 m 24. 216 cm²
25. 8 min. $16{\cdot}8$ sec. 26. 1 cm 27. $16{\cdot}01$ cm
28. 80,080 cm³.

SIMILAR TRIANGLES

Exercise 6 (a)

1. (*a*) 6 cm (*b*) AE=$4{\cdot}5$ cm, EC=3 cm
2. (*a*) 28 mm (*b*) $2{\cdot}8$ cm
4. PR=12 cm, QR=$10{\cdot}4$ cm
7. (*a*) AP=15 mm, PB=9 mm, AQ=$17{\cdot}5$ mm, QC=$10{\cdot}5$ mm
   (*b*) AP=60 mm, PB=36 mm, AQ=70 mm, QC=42 mm

Exercise 6 (b)

3. (*a*) $\triangle$ AEB ~ $\triangle$ DEC (*b*) 6 cm
5. $13{\cdot}5$ cm 6. $x=6\frac{2}{3}$ cm, $y=3\frac{3}{4}$ cm

Exercise 6 (c)

2. 3 : 4 3. $11{\cdot}2$ cm 7. (*a*) 5 (*b*) 1 : 9

Exercise 6 (d)

1. $\sqrt{l^2-h^2}$ metres 2. 12 m 4. 2 cm 7. 20 cm
8. 62 dm.

Review Exercise V

1. (*a*) True (*b*) True (*c*) True
2. (*a*) 9 : 4 (*b*) proportional (*c*) squares (*d*) parallel
   (*e*) 16 cm (*f*) similar (*g*) $1{\cdot}8$ cm (*h*) need not
3. (*i*) $\angle ACB=\angle ECD$ and $\frac{BC}{AC}=\frac{CE}{CD}$ (*ii*) 40° (*iii*) $2x$
4. 31 cm 5. 4 cm 10. (*i*) $12{\cdot}5$ cm (*ii*) $2{\cdot}5\,x$

CIRCLES

Exercise 7 (a)

1. Zero 2. Two 3. No. 4. Yes
5. Yes 6. Diameter=2×Radius
7. When it passes through the centre of the circle 8. Twice the radius

Exercise 7 (c)

2. 4 cm 3. $2{\cdot}5$ cm 4. (*i*) 1 cm (*ii*) 7 cm
5. (*a*) $3{\cdot}5$ cm (*b*) 5 mm. 6. $4\frac{1}{6}$ cm.
9. 2 cm, 8 cm ; O, P and A are collinear.

Exercise 7 (e)

5. 2 cm, 2 cm

Exercise 7 (f)

1. $\angle$ POR$=136°$ 2. $58°$ 5. (*a*) $65°$ (*b*) $130°$
6. $\angle$OBA$=50°$, $\angle$OAC$=40°$, 7. $140°$ 9. (*a*) $62°$ (*b*) $28°$

Exercise 7 (g)

2. $66°$ 5. $\angle$MAN$=110°$, $\angle$MON$=120°$ 6. (*a*) $25°$ (*b*) $30°$ (*c*) $55°$

Exercise 7 (k)

5. $37°$ 6. $\angle$BAD$=110°$, $\angle$BCD$=70°$ 7. $80°$
8. (*i*) $55°$ (*ii*) $35°$ 9. (*a*) $50°$ (*b*) $40°$ (*c*) $90°$
10. (*a*) $27°$ (*b*) $83°$ (*c*) $70°$

Review Exercise VI

1. (*a*) False (*b*) True (*c*) True (*d*) True
2. (*a*) equal (*b*) right angle (*c*) supplementary (*d*) perpendicular
   (*e*) equidistant (*f*) equal angles (*g*) one (*h*) equal
3. (*a*) $40°$ (*b*) $55°$ (*c*) $80°$ (*d*) 8 cm
6. $\angle$RNM$=119°$, $\angle$NRM$=32°$ 7. $50°$
10. $\angle$PRB$=35°$, $\angle$PBR$=115°$, $\angle$BPR$=30°$
12. (*ii*) $50°$.

TANGENT TO A CIRCLE

Exercise 8 (a)

6. 4 cm 7. 8 cm 9. (*a*) 10 cm (*b*) 8·7 cm (*c*) 43·5 cm^2

Exercise 8 (b)

3. 4 cm 4. 5 cm

Exercise 8 (c)

3. $65°$ 4. $33°$, $81°$, $66°$

Exercise 8 (d)

1. 8 cm

Exercise 8 (e)

1. (*a*) 4·2 cm (*b*) 6 mm 2. 25 mm, 35 mm, 30 mm
7. 7 mm, 10 mm, 12 mm.

Review Exercise VII

1. (*a*) perpendicular (*b*) equal (*c*) common tangent (*d*) centres
   (*e*) $r+s$ (*f*) direct (*g*) 6 cm (*h*) $60°$
   (*i*) $120°$ (*j*) 12 cm

GEOMETRICAL CONSTRUCTIONS

Exercise 9 (a)

1. (*i*) Circumcentre (*ii*) Equal (*iii*) Yes (*iv*) Orthocentre
2. (*i*) In-centre (*ii*) OR$=$OQ (*iii*) $\angle$ACO$=\angle$BCO
4. 1·6 cm (nearly) 5. 1·6 cm (nearly) 8. 8 mm (nearly)
9. 5 mm (nearly) 10. 1·2 cm (nearly) ; 2·3 cm 11. 1·3 cm

Exercise 9 (c)

1. 6·9 cm
2. 9·8 cm nearly
3. Each 5·4 cm (nearly)
4. Each 2 cm (nearly)
5. 4·4 cm each
6. 7·4 cm (nearly).

Review Exercise VIII

1. (*a*) False (*b*) True (*c*) True (*d*) True

STATISTICS

Exercise 10 (a)

1. Rs. 780
2. 34·7°C
3. 69·5 kg
4. 164·5 cm
5. 4
6. $y=13-x$
7. 35
8. 153 cm
9. (*a*) 6·6

Exercise 10 (b)

1. 3·2 marks
2. 2·5 marks
3. Rs. 10·54
4. 14·19 years
5. 61·6 marks
6. 159 cm
7. Rs. 182

Exercise 10 (c)

1. 31·28 years
2. Rs. 28
3. 33·33 years
4. 28·1 marks
5. 25·4
6. 33·3 years
7. 18·45 years

Exercise 10 (d)

1. 66
2. 4 kg
3. 62·7 cm
4. 21·8
5. 3·16 litres
6. 13·01 years
7. 19·6
8. 14·26 years
9. 45·6
10. Rs. 145
11. 20·27
12. Rs. 56·31
13. 154·5 cm
14. Rs. 112·50
15. Rs. 28
16. 38·67

Exercise 10 (e)

1. Median
2. 46
3. 5
4. 4·5
5. 13·5
6. 58 ; 58
7. {1, 2, 3, 4, 10} ; {1, 2, 3, 5, 9}

Exercise 10 (f)

1. (*a*) population (*b*) 1000 (*c*) barometers (*d*) time
3. 125
4. 110.

Review Exercise IX

1. (*a*) 12 (*b*) 998 (*c*) 35 (*d*) 5·5
   (*e*) 5 (*f*) 3 (*g*) 65
2. 10
3. 4
4. 156 cm
5. 6·14
6. $\dfrac{pq+1000s}{1000\,(p+r)}$ kg
7. 207·54 cm
8. 60·7 years
9. 22
10. 174
11. Rs. 472
12. Rs. 229·15.

TRIGONOMETRY

Exercise 11 (c)

1. No
2. No
3. No
4. Yes
5. No
6. $\theta=60°$
7. $\theta=45°$
8. $\theta=30°$
9. $\theta=60°$
10. $\theta=60°$
11. $\theta=60°$
12. $\theta=45°$

Exercise 11 (d)

1. $\cos 9°+\cot 9°$
2. 0
3. 1
4. 1
5. 1
9. 1
12. 1 : 4

Exercise 11 (e)

1. 0·5736	**2.** 0·9613	**3.** 0·7536	**4.** 0·8192
5. 1·192	**6.** 2·203	**7.** 2·177	**8.** 0·8511
9. 2·381	**10.** 0·2943	**11.** 1·6029	**12.** 0·4337
13. 1·6433	**14.** 1·2467	**15.** 2·167	

Exercise 11 (f)

1. 1·732	**2.** 1·9022	**3.** 1·1680	**4.** 1·5308
5. 0·7626	**6.** 6·3 cm	**7.** 19·6 cm	**8.** 46·05 cm^2
9. 4·1 : 4	**10.** 3·95 cm^2		

Exercise 11 (g)

1. 61·6 m	**2.** 2·344 km	**3.** 60·22 m	**4.** 200 m
5. 15·91 m	**6.** 7·5 m	**7.** 683 m	**8.** 18·59 m
9. 285·75 m	**10.** 7·098 km	**11.** 42·55 m ; 20·75 m	
12. 37·1 m ; 1271·6 m.			

Review Exercise X

1. (*a*) 0	(*b*) 1	(*c*) 0	(*d*) 1
(*e*) 0	(*f*) 1		
2. (*a*) 1	(*b*) 0		
8. 13·25 m	**9.** 26·1 m	**12.** 77·62 m	**13.** 23·79 m

TEST PAPERS

Test Paper 1.

1. (*a*) 0	(*b*) 3 cm	(*c*) 114°	(*d*) 7
(*e*) P(*a*)=0.			
2. (*a*) 0 ; $-\sqrt{3}$	(*b*) No	(*c*) Inconsistent	(*d*) $2x-7$
(*e*) $(x-4)(x+3)$.			
3. (*b*) 6 cm	(*c*) 199·73 metres.		
6. (*a*) $x=\pm\sqrt{3}$	(*b*) $\dfrac{4(x^2+1)^2}{(x^2-1)^2}$	(*c*) 8	
7. (*a*) 2·016 m^2	(*b*) 126	(*c*) 10 m	
8. (*a*) Rs. 1500	(*b*) 19·2 years.		

Test Paper 2.

1. (*a*) bisects	(*b*) 10 cm	(*c*) 1	(*d*) +3, −3
(*e*) consistent.			
2. (*a*) 1 ; $-\frac{3}{4}$	(*b*) $k=\pm 6$	(*c*) $(2x-7)^2(x+3)(3x+4)$	
(*d*) No, the two circles intersect each other.		(*e*) 40°, 140°	
3. (*c*) 36·05 m			
5. (*a*) 2, $\frac{1}{2}$, 3, $\frac{1}{3}$	(*b*) 6 yrs., 36 yrs.	(*c*) $x=2$; $y=-3$	
7. (*a*) 462 m^3	(*b*) 42 cm^2	(*c*) 66 cm^2	
8. (*a*) 49·67 kg	(*b*) 166·35		

Test Paper 3.

1. (*a*) perpendicular (*b*) radii (*c*) 50° (*d*) ϕ
   (*e*) $\frac{1}{3}\pi r^2 h$
2. (*a*) Has no zeros (*b*) $x^2-x-6=0$ (*c*) $\dfrac{8x^7-\sqrt{3x}}{x^7-\sqrt{2x}}$ (*d*) 8 cm
   (*e*) many
3. (*a*) $\dfrac{2-\sqrt{2}}{3-\sqrt{2}}, \dfrac{2+\sqrt{2}}{3+\sqrt{2}}$ (*b*) 5, 6, 7 (*c*) zero
6. (*a*) 55 cm ; 577·5 cm² (*b*) 74·18 m³ ; 80·54 m² (*c*) 2 : 3
7. (*a*) 1 (*c*) 126·21 m
8. (*a*) Rs. 145 (*b*) 151.

Test Paper 4.

1. (*a*) equidistant (*b*) 1 (*c*) 40° (*d*) equal
   (*e*) three
2. (*a*) $k=\pm 4$ (*b*) $\dfrac{1}{2x-1}$ (*c*) 9 : 4 (*d*) when inconsistent
   (*e*) $3\pi x^2$ cm²
3. (*a*) $-2, \dfrac{-19}{3}$ (*b*) $x=-3,\ y=1$ (*c*) 10 m, 7 m
5. (*c*) 3·19 m
7. (*a*) 10·26 cm² ; 615·44 cm² (*b*) 1232 cm² (*c*) 16
8. (*a*) $f_1=28, f_2=24$ (*b*) 42·6 quintals per hectare

Test Paper 5.

1. (*a*) 65° (*b*) cyclic (*c*) 1 (*d*) $x=4,\ y=1$
   (*e*) real and equal
2. (*a*) 3 (*b*) $\dfrac{x^3+x^2+4x+2}{x+1}$ (*c*) 48°, 132°
   (*d*) 6 cm (*e*) $y=13-x$ 3. (*c*) 71·66 m
6. (*a*) Rs. 9·18 (*b*) 52·3
7. (*a*) $x=4, 5$ (*b*) $17x^2-20x+3=0$ (*c*) $x=5,\ y=1$
8. (*a*) 2541 (*b*) 3·21 cm² (*c*) 4·67 m.

□ □

APPENDIX
Logarithm Tables
LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						5	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	0414	0453	0492	0531	0569						4	8	12	16	20	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	18	22	26	29	33
12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	6	10	13	16	19	23	26	29
						1303	1335	1367	1399	1430	3	7	10	13	16	19	22	25	29
14	1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
						1614	1644	1673	1703	1732	3	6	9	12	14	17	20	23	26
15	1761	1790	1818	1847	1875						3	6	9	11	14	17	20	23	26
						1903	1931	1959	1987	2014	3	6	8	11	14	17	19	22	25
16	2041	2068	2095	2122	2148						3	6	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
17	2304	2330	2355	2380	2405						3	5	8	10	13	15	18	20	23
						2430	2455	2480	2504	2529	3	5	8	10	12	15	17	20	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	17	19	21
						2672	2695	2718	2742	2765	2	4	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1 2 3	3 4 5	6 7 8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1 2 3	3 4 5	6 7 8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1 2 2	3 4 5	6 7 7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1 2 2	3 4 5	6 6 7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1 2 2	3 4 5	6 6 7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1 2 2	3 4 5	5 6 7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1 2 2	3 4 5	5 6 7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1 2 2	3 4 5	5 6 7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1 1 2	3 4 4	5 6 7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1 1 2	3 4 4	5 6 7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1 1 2	3 4 4	5 6 6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1 1 2	3 4 4	5 6 6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1 1 2	3 3 4	5 6 6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1 1 2	3 3 4	5 5 6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1 1 2	3 3 4	5 5 6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1 1 2	3 3 4	5 5 6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1 1 2	3 3 4	5 5 6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1 1 2	3 3 4	5 5 6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1 1 2	3 3 4	5 5 6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1 1 2	2 3 4	4 5 6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1 1 2	2 3 4	4 5 6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1 1 2	2 3 4	4 5 5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1 1 2	2 3 4	4 5 5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1 1 2	2 3 4	4 5 5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1 1 2	2 3 4	4 5 5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1 1 2	2 3 3	4 5 5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1 1 2	2 3 3	4 5 5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1 1 2	2 3 3	4 4 5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1 1 2	2 3 3	4 4 5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1 1 2	2 3 3	4 4 5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1 1 2	2 3 3	4 4 5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1 1 2	2 3 3	4 4 5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1 1 2	2 3 3	4 4 5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1 1 2	2 3 3	4 4 5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1 1 2	2 3 3	4 4 5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1 1 2	2 3 3	4 4 5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1 1 2	2 3 3	4 4 5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0 1 1	2 2 3	3 4 4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0 1 1	2 2 3	3 4 4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0 1 1	2 2 3	3 4 4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0 1 1	2 2 3	3 4 4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0 1 1	2 2 3	3 4 4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0 1 1	2 2 3	3 4 4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0 1 1	2 2 3	3 4 4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0 1 1	2 2 3	3 4 4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0 1 1	2 2 3	3 4 4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0 1 1	2 2 3	3 4 4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0 1 1	2 2 3	3 4 4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0 1 1	2 2 3	3 4 4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0 1 1	2 2 3	3 3 4

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	123	456	789
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0 0 1	1 1 1	2 2 2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0 0 1	1 1 1	2 2 2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0 0 1	1 1 1	2 2 2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0 0 1	1 1 1	2 2 2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0 1 1	1 1 2	2 2 2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0 1 1	1 1 2	2 2 2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0 1 1	1 1 2	2 2 2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0 1 1	1 1 2	2 2 2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0 1 1	1 1 2	2 2 3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0 1 1	1 1 2	2 2 3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0 1 1	1 1 2	2 2 3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0 1 1	1 2 2	2 2 3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0 1 1	1 2 2	2 2 3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0 1 1	1 2 2	2 3 3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0 1 1	1 2 2	2 3 3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0 1 1	1 2 2	2 3 3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0 1 1	1 2 2	2 3 3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0 1 1	1 2 2	2 3 3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0 1 1	1 2 2	2 3 3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0 1 1	1 2 2	3 3 3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0 1 1	1 2 2	3 3 3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0 1 1	2 2 2	3 3 3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0 1 1	2 2 2	3 3 3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0 1 1	2 2 2	3 3 4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0 1 1	2 2 2	3 3 4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0 1 1	2 2 2	3 3 4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0 1 1	2 2 3	3 3 4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0 1 1	2 2 3	3 3 4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0 1 1	2 2 3	3 4 4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0 1 1	2 2 3	3 4 4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0 1 1	2 2 3	3 4 4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0 1 1	2 2 3	3 4 4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0 1 1	2 2 3	3 4 4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0 1 1	2 2 3	3 4 4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1 1 2	2 3 3	4 4 5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2285	1 1 2	2 3 3	4 4 5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1 1 2	2 3 3	4 4 5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1 1 2	2 3 3	4 4 5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1 1 2	2 3 3	4 4 5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1 1 2	2 3 3	4 5 5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1 1 2	2 3 4	4 5 5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1 1 2	2 3 4	4 5 5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1 1 2	2 3 4	4 5 6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1 1 2	3 3 4	4 5 6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1 1 2	3 3 4	4 5 6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1 1 2	3 3 4	5 5 6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1 1 2	3 3 4	5 5 6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1 1 2	3 3 4	5 5 6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1 1 2	3 3 4	5 6 6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1 1 2	3 3 4	5 6 6

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1 1 2	3 4 4	5 6 7
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1 2 2	3 4 5	5 6 7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1 2 2	3 4 5	5 6 7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1 2 2	3 4 5	6 6 7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1 2 2	3 4 5	6 6 7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1 2 2	3 4 5	6 7 7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1 2 3	3 4 5	6 7 8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1 2 3	3 4 5	6 7 8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1 2 3	4 4 5	6 7 8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1 2 3	4 5 5	6 7 8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1 2 3	4 5 6	6 7 8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1 2 3	4 5 6	7 8 9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1 2 3	4 5 6	7 8 9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1 2 3	4 5 6	7 8 9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1 2 3	4 5 6	7 8 9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1 2 3	4 5 6	7 8 9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1 2 3	4 5 6	7 9 10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1 2 3	4 5 7	8 9 10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1 2 3	4 6 7	8 9 10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1 2 3	5 6 7	8 9 10
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1 2 4	5 6 7	8 9 11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1 2 4	5 6 7	8 10 11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1 2 4	5 6 7	9 10 11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1 3 4	5 6 8	9 10 11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1 3 4	5 6 8	9 10 12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1 3 4	5 7 8	9 10 12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1 3 4	5 7 8	9 11 12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1 3 4	5 7 8	10 11 12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1 3 4	6 7 8	10 11 13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1 3 4	6 7 9	10 11 13
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1 3 4	6 7 9	10 12 13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2 3 5	6 8 9	11 12 14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2 3 5	6 8 9	11 12 14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2 3 5	6 8 9	11 13 14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2 3 5	6 8 10	11 13 15
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2 3 5	7 8 10	12 13 15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2 3 5	7 8 10	12 13 15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2 3 5	7 9 10	12 14 16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2 4 5	7 9 11	12 14 16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2 4 5	7 9 11	13 14 16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2 4 6	7 9 11	13 15 17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2 4 6	8 9 11	13 15 17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2 4 6	8 10 12	14 15 17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2 4 6	8 10 12	14 16 18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2 4 6	8 10 12	14 16 18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2 4 6	8 10 12	15 17 19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2 4 6	8 11 13	15 17 19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2 4 7	9 11 13	15 17 20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2 4 7	9 11 13	16 18 20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2 5 7	9 11 14	16 18 20

TABLE I

TRIGONOMETRIC RATIOS — DEGREES AND MINUTES

↓→	sin	cos	tan	cot	sec	cosec	
0°00′	0.0000	1.000	0.0000	—	1.000	—	90°00′
10′	0.0029	1.000	0.0029	343.8	1.000	343.8	89°50′
20′	0.0058	1.000	0.0058	171.9	1.000	171.9	40′
30′	0.0087	1.000	0.0087	114.6	1.000	114.6	30′
40′	0.0116	0.9999	0.0116	85.94	1.000	85.95	20′
0°50′	0.0145	0.9999	0.0145	68.75	1.000	68.76	10′
1°00′	0.0175	0.9998	0.0175	57.29	1.000	57.30	89°00′
10′	0.0204	0.9998	0.0204	49.10	1.000	49.11	88°50′
20′	0.0233	0.9997	0.0233	42.96	1.000	42.98	40′
30′	0.0262	0.9997	0.0262	38.19	1.000	38.20	30′
40′	0.0291	0.9996	0.0291	34.37	1.000	34.38	20′
1°50′	0.0320	0.9995	0.0320	31.24	1.001	31.26	10′
2°00′	0.0349	0.9994	0.0349	28.64	1.001	28.65	88°00′
10′	0.0378	0.9993	0.0378	26.43	1.001	26.45	87°50′
20′	0.0407	0.9992	0.0407	24.54	1.001	24.56	40′
30′	0.0436	0.9990	0.0437	22.90	1.001	22.93	30′
40′	0.0465	0.9989	0.0466	21.47	1.001	21.49	20′
2°50′	0.0494	0.9988	0.0495	20.21	1.001	20.23	10′
3°00′	0.0523	0.9986	0.0524	19.08	1.001	19.11	87°00′
10′	0.0552	0.9985	0.0553	18.07	1.002	18.10	86°50′
20′	0.0581	0.9983	0.0582	17.17	1.002	17.20	40′
30′	0.0610	0.9981	0.0612	16.35	1.002	16.38	30′
40′	0.0640	0.9980	0.0641	15.60	1.002	15.64	20′
3°50′	0.0669	0.9978	0.0670	14.92	1.002	14.96	10′
4°00′	0.0698	0.9976	0.0699	14.30	1.002	14.34	86°00′
10′	0.0727	0.9974	0.0729	13.73	1.003	13.76	85°50′
20′	0.0756	0.9971	0.0758	13.20	1.003	13.23	40′
30′	0.0785	0.9969	0.0787	12.71	1.003	12.75	30′
40′	0.0814	0.9967	0.0816	12.25	1.003	12.29	20′
4°50′	0.0843	0.9964	0.0846	11.83	1.004	11.87	10′
5°00′	0.0872	0.9962	0.0875	11.43	1.004	11.47	85°00′
10′	0.0901	0.9959	0.0904	11.06	1.004	11.10	84°50′
20′	0.0929	0.9957	0.0934	10.71	1.004	10.76	40′
30′	0.0958	0.9954	0.0963	10.39	1.005	10.43	30′
40′	0.0987	0.9951	0.0992	10.08	1.005	10.13	20′
5°50′	0.1016	0.9948	0.1022	9.788	1.005	9.839	10′
6°00′	0.1045	0.9945	0.1051	9.514	1.006	9.567	84°00′
10′	0.1074	0.9942	0.1080	9.255	1.006	9.309	83°50′
20′	0.1103	0.9939	0.1110	9.010	1.006	9.065	40′
30′	0.1132	0.9936	0.1139	8.777	1.006	8.834	30′
40′	0.1161	0.9932	0.1169	8.556	1.007	8.614	20′
6°50′	0.1190	0.9929	0.1198	8.345	1.007	8.405	10′
7°00′	0.1219	0.9925	0.1228	8.144	1.008	8.206	83°00′
	cos	sin	cot	tan	cosec	sec	←↑

TABLE I (CONTINUED)

↓→	sin	cos	tan	cot	sec	cosec	
7°00′	0.1219	0.9925	0.1228	8.144	1.008	8.206	83°00′
10′	0.1248	0.9922	0.1257	7.953	1.008	8.016	82°50′
20′	0.1276	0.9918	0.1287	7.770	1.008	7.834	40′
30′	0.1305	0.9914	0.1317	7.596	1.009	7.661	30′
40′	0.1334	0.9911	0.1346	7.429	1.009	7.496	20′
7°50′	0.1363	0.9907	0.1376	7.269	1.009	7.337	10′
8°00′	0.1392	0.9903	0.1405	7.115	1.010	7.185	82°00′
10′	0.1421	0.9899	0.1435	6.968	1.010	7.040	81°50′
20′	0.1449	0.9894	0.1465	6.827	1.011	6.900	40′
30′	0.1478	0.9890	0.1495	6.691	1.011	6.765	30′
40′	0.1507	0.9886	0.1524	6.561	1.012	6.636	20′
8°50′	0.1536	0.9881	0.1554	6.435	1.012	6.512	10′
9°00′	0.1564	0.9877	0.1584	6.314	1.012	6.392	81°00′
10′	0.1593	0.9872	0.1614	6.197	1.013	6.277	80°50′
20′	0.1622	0.9868	0.1644	6.084	1.013	6.166	40′
30′	0.1650	0.9863	0.1673	5.976	1.014	6.059	30′
40′	0.1679	0.9858	0.1703	5.871	1.014	5.955	20′
9°50′	0.1708	0.9853	0.1733	5.769	1.015	5.855	10′
10°00′	0.1736	0.9848	0.1763	5.671	1.015	5.759	80°00′
10′	0.1765	0.9843	0.1793	5.576	1.016	5.665	79°50′
20′	0.1794	0.9838	0.1823	5.485	1.016	5.575	40′
30′	0.1822	0.9833	0.1853	5.396	1.017	5.487	30′
40′	0.1851	0.9827	0.1883	5.309	1.018	5.403	20′
10°50′	0.1880	0.9822	0.1914	5.226	1.018	5.320	10′
11°00′	0.1908	0.9816	0.1944	5.145	1.019	5.241	79°00′
10′	0.1937	0.9811	0.1974	5.066	1.019	5.164	78°50′
20′	0.1965	0.9805	0.2004	4.989	1.020	5.089	40′
30′	0.1994	0.9799	0.2035	4.915	1.020	5.016	30′
40′	0.2022	0.9793	0.2065	4.843	1.021	4.945	20′
11°50′	0.2051	0.9787	0.2095	4.773	1.022	4.876	10′
12°00′	0.2079	0.9781	0.2126	4.705	1.022	4.810	78°00′
10′	0.2108	0.9775	0.2156	4.638	1.023	4.745	77°50′
20′	0.2136	0.9769	0.2186	4.574	1.024	4.682	40′
30′	0.2164	0.9763	0.2217	4.511	1.024	4.620	30′
40′	0.2193	0.9757	0.2247	4.449	1.025	4.560	20′
12°50′	0.2221	0.9750	0.2278	4.390	1.026	4.502	10′
13°00′	0.2250	0.9744	0.2309	4.331	1.026	4.445	77°00′
10′	0.2278	0.9737	0.2339	4.275	1.027	4.390	76°50′
20′	0.2306	0.9730	0.2370	4.219	1.028	4.336	40′
30′	0.2334	0.9724	0.2401	4.165	1.028	4.284	30′
40′	0.2363	0.9717	0.2432	4.113	1.029	4.232	20′
13°50′	0.2391	0.9710	0.2462	4.061	1.030	4.182	10′
14°00′	0.2419	0.9703	0.2493	4.011	1.031	4.134	76°00′
	cos	sin	cot	tan	cosec	sec	←↑

TABLE I (CONTINUED)

↓→	sin	cos	tan	cot	sec	cosec	
14°00′	0.2419	0.9703	0.2493	4.011	1.031	4.134	76°00′
10′	0.2447	0.9696	0.2524	3.962	1.031	4.086	75°50′
20′	0.2476	0.9689	0.2555	3.914	1.032	4.039	40′
30′	0.2504	0.9681	0.2586	3.867	1.033	3.994	30′
40′	0.2532	0.9674	0.2617	3.821	1.034	3.950	20′
14°50′	0.2560	0.9667	0.2648	3.776	1.034	3.906	10′
15°00′	0.2588	0.9659	0.2679	3.732	1.035	3.864	75°00′
10′	0.2616	0.9652	0.2711	3.689	1.036	3.822	74°50′
20′	0.2644	0.9644	0.2742	3.647	1.037	3.782	40′
30′	0.2672	0.9636	0.2773	3.606	1.038	3.742	30′
40′	0.2700	0.9628	0.2805	3.566	1.039	3.703	20′
15°50′	0.2728	0.9621	0.2836	3.526	1.039	3.665	10′
16°00′	0.2756	0.9613	0.2867	3.487	1.040	3.628	74°00′
10′	0.2784	0.9605	0.2899	3.450	1.041	3.592	73°50′
20′	0.2812	0.9596	0.2931	3.412	1.042	3.556	40′
30′	0.2840	0.9588	0.2962	3.376	1.043	3.521	30′
40′	0.2868	0.9580	0.2994	3.340	1.044	3.487	20′
16°50′	0.2896	0.9572	0.3026	3.305	1.045	3.453	10′
17°00′	0.2924	0.9563	0.3057	3.271	1.046	3.420	73°00′
10′	0.2952	0.9555	0.3089	3.237	1.047	3.388	72°50′
20′	0.2979	0.9546	0.3121	3.204	1.048	3.356	40′
30′	0.3007	0.9537	0.3153	3.172	1.049	3.326	30′
40′	0.3035	0.9528	0.3185	3.140	1.049	3.295	20′
17°50′	0.3062	0.9520	0.3217	3.108	1.050	3.265	10′
18°00′	0.3090	0.9511	0.3249	3.078	1.051	3.236	72°00′
10′	0.3118	0.9502	0.3281	3.047	1.052	3.207	71°50′
20′	0.3145	0.9492	0.3314	3.018	1.053	3.179	40′
30′	0.3173	0.9483	0.3346	2.989	1.054	3.152	30′
40′	0.3201	0.9474	0.3378	2.960	1.056	3.124	20′
18°50′	0.3228	0.9465	0.3411	2.932	1.057	3.098	10′
19°00′	0.3256	0.9455	0.3443	2.904	1.058	3.072	71°00′
10′	0.3283	0.9446	0.3476	2.877	1.059	3.046	70°50′
20′	0.3311	0.9436	0.3508	2.850	1.060	3.021	40′
30′	0.3338	0.9426	0.3541	2.824	1.061	2.996	30′
40′	0.3365	0.9417	0.3574	2.798	1.062	2.971	20′
19°50′	0.3393	0.9407	0.3607	2.773	1.063	2.947	10′
20°00′	0.3420	0.9397	0.3640	2.747	1.064	2.924	70°00′
10′	0.3448	0.9387	0.3673	2.723	1.065	2.901	69°50′
20′	0.3475	0.9377	0.3706	2.699	1.066	2.878	40′
30′	0.3502	0.9367	0.3739	2.675	1.068	2.855	30′
40′	0.3529	0.9356	0.3772	2.651	1.069	2.833	20′
20°50′	0.3557	0.9346	0.3805	2.628	1.070	2.812	10′
21°00′	0.3584	0.9336	0.3839	2.605	1.071	2.790	69°00′
	cos	sin	cot	tan	cosec	sec	←↑

TABLE I (CONTINUED)

↳	sin	cos	tan	cot	sec	cosec	
21°00′	0.3584	0.9336	0.3839	2.605	1.071	2.790	69°00′
10′	0.3611	0.9325	0.3872	2.583	1.072	2.769	68°50′
20′	0.3638	0.9315	0.3906	2.560	1.074	2.749	40′
30′	0.3665	0.9304	0.3939	2.539	1.075	2.729	30′
40′	0.3692	0.9293	0.3973	2.517	1.076	2.709	20′
21°50′	0.3719	0.9283	0.4006	2.496	1.077	2.689	10′
22°00′	0.3746	0.9272	0.4040	2.475	1.079	2.669	68°00′
10′	0.3773	0.9261	0.4074	2.455	1.080	2.650	67°50′
20′	0.3800	0.9250	0.4108	2.434	1.081	2.632	40′
30′	0.3827	0.9239	0.4142	2.414	1.082	2.613	30′
40′	0.3854	0.9228	0.4176	2.394	1.084	2.595	20′
22°50′	0.3881	0.9216	0.4210	2.375	1.085	2.577	10′
23°00′	0.3907	0.9205	0.4245	2.356	1.086	2.559	67°00′
10′	0.3934	0.9194	0.4279	2.337	1.088	2.542	66°50′
20′	0.3961	0.9182	0.4314	2.318	1.089	2.525	40′
30′	0.3987	0.9171	0.4348	2.300	1.090	2.508	30′
40′	0.4014	0.9159	0.4383	2.282	1.092	2.491	20′
23°50′	0.4041	0.9147	0.4417	2.264	1.093	2.475	10′
24°00′	0.4067	0.9135	0.4452	2.246	1.095	2.459	66°00′
10′	0.4094	0.9124	0.4487	2.229	1.096	2.443	65°50′
20′	0.4120	0.9112	0.4522	2.211	1.097	2.427	40′
30′	0.4147	0.9100	0.4557	2.194	1.099	2.411	30′
40′	0.4173	0.9088	0.4592	2.177	1.100	2.396	20′
24°50′	0.4200	0.9075	0.4628	2.161	1.102	2.381	10′
25°00′	0.4226	0.9063	0.4663	2.145	1.103	2.366	65°00′
10′	0.4253	0.9051	0.4699	2.128	1.105	2.352	64°50′
20′	0.4279	0.9038	0.4734	2.112	1.106	2.337	40′
30′	0.4305	0.9026	0.4770	2.097	1.108	2.323	30′
40′	0.4331	0.9013	0.4806	2.081	1.109	2.309	20′
25°50′	0.4358	0.9001	0.4841	2.066	1.111	2.295	10′
26°00′	0.4384	0.8988	0.4877	2.050	1.113	2.281	64°00′
10′	0.4410	0.8975	0.4913	2.035	1.114	2.268	63°50′
20′	0.4436	0.8962	0.4950	2.020	1.116	2.254	40′
30′	0.4462	0.8949	0.4986	2.006	1.117	2.241	30′
40′	0.4488	0.8936	0.5022	1.991	1.119	2.228	20′
26°50′	0.4514	0.8923	0.5059	1.977	1.121	2.215	10′
27°00′	0.4540	0.8910	0.5095	1.963	1.122	2.203	63°00′
10′	0.4566	0.8897	0.5132	1.949	1.124	2.190	62°50′
20′	0.4592	0.8884	0.5169	1.935	1.126	2.178	40′
30′	0.4617	0.8870	0.5206	1.921	1.127	2.166	30′
40′	0.4643	0.8857	0.5243	1.907	1.129	2.154	20′
27°50′	0.4669	0.8843	0.5280	1.894	1.131	2.142	10′
28°00′	0.4695	0.8829	0.5317	1.881	1.133	2.130	62°00′
	cos	sin	cot	tan	cosec	sec	↲

TABLE I (CONTINUED)

↓→	sin	cos	tan	cot	sec	cosec	
28°00′	0.4695	0.8829	0.5317	1.881	1.133	2.130	62°00′
10′	0.4720	0.8816	0.5354	1.868	1.134	2.118	61°50′
20′	0.4746	0.8802	0.5392	1.855	1.136	2.107	40′
30′	0.4772	0.8788	0.5430	1.842	1.138	2.096	30′
40′	0.4797	0.8774	0.5467	1.829	1.140	2.085	20′
28°50′	0.4823	0.8760	0.5505	1.816	1.142	2.074	10′
29°00′	0.4848	0.8746	0.5543	1.804	1.143	2.063	61°00′
10′	0.4874	0.8732	0.5581	1.792	1.145	2.052	60°50′
20′	0.4899	0.8718	0.5619	1.780	1.147	2.041	40′
30′	0.4924	0.8704	0.5658	1.767	1.149	2.031	30′
40′	0.4950	0.8689	0.5696	1.756	1.151	2.020	20′
29°50′	0.4975	0.8675	0.5735	1.744	1.153	2.010	10′
30°00′	0.5000	0.8660	0.5774	1.732	1.155	2.000	60°00′
10′	0.5025	0.8646	0.5812	1.720	1.157	1.990	59°50′
20′	0.5050	0.8631	0.5851	1.709	1.159	1.980	40′
30′	0.5075	0.8616	0.5890	1.698	1.161	1.970	30′
40′	0.5100	0.8601	0.5930	1.686	1.163	1.961	20′
30°50′	0.5125	0.8587	0.5969	1.675	1.165	1.951	10′
31°00′	0.5150	0.8572	0.6009	1.664	1.167	1.942	59°00′
10′	0.5175	0.8557	0.6048	1.653	1.169	1.932	58°50′
20′	0.5200	0.8542	0.6088	1.643	1.171	1.923	40′
30′	0.5225	0.8526	0.6128	1.632	1.173	1.914	30′
40′	0.5250	0.8511	0.6168	1.621	1.175	1.905	20′
31°50′	0.5275	0.8496	0.6208	1.611	1.177	1.896	10′
32°00′	0.5299	0.8480	0.6249	1.600	1.179	1.887	58°00′
10′	0.5324	0.8465	0.6289	1.590	1.181	1.878	57°50′
20′	0.5348	0.8450	0.6330	1.580	1.184	1.870	40′
30′	0.5373	0.8434	0.6371	1.570	1.186	1.861	30′
40′	0.5398	0.8418	0.6412	1.560	1.188	1.853	20′
32°50′	0.5422	0.8403	0.6453	1.550	1.190	1.844	10′
33°00′	0.5446	0.8387	0.6494	1.540	1.192	1.836	57°00′
10′	0.5471	0.8371	0.6536	1.530	1.195	1.828	56°50′
20′	0.5495	0.8355	0.6577	1.520	1.197	1.820	40′
30′	0.5519	0.8339	0.6619	1.511	1.199	1.812	30′
40′	0.5544	0.8323	0.6661	1.501	1.202	1.804	20′
33°50′	0.5568	0.8307	0.6703	1.492	1.204	1.796	10′
34°00′	0.5592	0.8290	0.6745	1.483	1.206	1.788	56°00′
10′	0.5616	0.8274	0.6787	1.473	1.209	1.781	55°50′
20′	0.5640	0.8258	0.6830	1.464	1.211	1.773	40′
30′	0.5664	0.8241	0.6873	1.455	1.213	1.766	30′
40′	0.5688	0.8225	0.6916	1.446	1.216	1.758	20′
34°50′	0.5712	0.8208	0.6959	1.437	1.218	1.751	10′
35°00′	0.5736	0.8192	0.7002	1.428	1.221	1.743	55°00′
	cos	sin	cot	tan	cosec	sec	←↑

TABLE I (CONTINUED)

↓→	sin	cos	tan	cot	sec	cosec	
35°00′	0.5736	0.8192	0.7002	1.428	1.221	1.743	55°00′
10′	0.5760	0.8175	0.7046	1.419	1.223	1.736	54°50′
20′	0.5783	0.8158	0.7089	1.411	1.226	1.729	40′
30′	0.5807	0.8141	0.7133	1.402	1.228	1.722	30′
40′	0.5831	0.8124	0.7177	1.393	1.231	1.715	20′
35°50′	0.5854	0.8107	0.7221	1.385	1.233	1.708	10′
36°00′	0.5878	0.8090	0.7265	1.376	1.236	1.701	54°00′
10′	0.5901	0.8073	0.7310	1.368	1.239	1.695	53°50′
20	0.5925	0.8056	0.7355	1.360	1.241	1.688	40′
30′	0.5948	0.8039	0.7400	1.351	1.244	1.681	30′
40′	0.5972	0.8021	0.7445	1.343	1.247	1.675	20′
36°50′	0.5995	0.8004	0.7490	1.335	1.249	1.668	10′
37°00′	0.6018	0.7986	0.7536	1.327	1.252	1.662	53°00′
10′	0.6041	0.7969	0.7581	1.319	1.255	1.655	52°50′
20′	0.6065	0.7951	0.7627	1.311	1.258	1.649	40′
30′	0.6088	0.7934	0.7673	1.303	1.260	1.643	30′
40′	0.6111	0.7916	0.7720	1.295	1.263	1.636	20′
37°50′	0.6134	0.7898	0.7766	1.288	1.266	1.630	10′
38°00′	0.6157	0.7880	0.7813	1.280	1.269	1.624	52°00′
10′	0.6180	0.7862	0.7860	1.272	1.272	1.618	51°50′
20′	0.6202	0.7844	0.7907	1.265	1.275	1.612	40′
30′	0.6225	0.7826	0.7954	1.257	1.278	1.606	30′
40′	0.6248	0.7808	0.8002	1.250	1.281	1.601	20′
38°50′	0.6271	0.7790	0.8050	1.242	1.284	1.595	10′
39°00′	0.6293	0.7771	0.8098	1.235	1.287	1.589	51°00′
10′	0.6316	0.7753	0.8146	1.228	1.290	1.583	50°50′
20′	0.6338	0.7735	0.8195	1.220	1.293	1.578	40′
30′	0.6361	0.7716	0.8243	1.213	1.296	1.572	30′
40′	0.6383	0.7698	0.8292	1.206	1.299	1.567	20′
39°50′	0.6406	0.7679	0.8342	1.199	1.302	1.561	10′
40°00′	0.6428	0.7660	0.8391	1.192	1.305	1.556	50°00′
10′	0.6450	0.7642	0.8441	1.185	1.309	1.550	49°50′
20′	0.6472	0.7623	0.8491	1.178	1.312	1.545	40′
30′	0.6494	0.7604	0.8541	1.171	1.315	1.540	30′
40′	0.6517	0.7585	0.8591	1.164	1.318	1.535	20′
40°50′	0.6539	0.7566	0.8642	1.157	1.322	1.529	10′
41°00′	0.6561	0.7547	0.8693	1.150	1.325	1.524	49°00′
10′	0.6583	0.7528	0.8744	1.144	1.328	1.519	48°50′
20′	0.6604	0.7509	0.8796	1.137	1.332	1.514	40′
30′	0.6626	0.7490	0.8847	1.130	1.335	1.509	30′
40′	0.6648	0.7470	0.8899	1.124	1.339	1.504	20′
41°50′	0.6670	0.7451	0.8952	1.117	1.342	1.499	10′
42°00′	0.6691	0.7431	0.9004	1.111	1.346	1.494	48°00′
	cos	sin	cot	tan	cosec	sec	←↑

TABLE I (CONTINUED)

↓→	sin	cos	tan	cot	sec	cosec	
42°00′	0.6691	0.7431	0.9004	1.111	1.346	1.494	48°00′
10′	0.6713	0.7412	0.9057	1.104	1.349	1.490	47°50′
20′	0.6734	0.7392	0.9110	1.098	1.353	1.485	40′
30′	0.6756	0.7373	0.9163	1.091	1.356	1.480	30′
40′	0.6777	0.7353	0.9217	1.085	1.360	1.476	20′
42°50′	0.6799	0.7333	0.9271	1.079	1.364	1.471	10′
43°00′	0.6820	0.7314	0.9325	1.072	1.367	1.466	47°00′
10′	0.6641	0.7294	0.9380	1.066	1.371	1.462	46°50′
20′	0.6862	0.7274	0.9435	1.060	1.375	1.457	40′
30′	0.6884	0.7254	0.9490	1.054	1.379	1.453	30′
40′	0.6905	0.7234	0.9545	1.048	1.382	1.448	20′
43°50′	0.6926	0.7214	0.9601	1.042	1.386	1.444	10′
44°00′	0.6947	0.7193	0.9657	1.036	1.390	1.440	46°00′
10′	0.6967	0.7173	0.9713	1.030	1.394	1.435	45°50′
20′	0.6988	0.7153	0.9770	1.024	1.398	1.431	40′
30′	0.7009	0.7133	0.9827	1.018	1.402	1.427	30′
40′	0.7030	0.7112	0.984	1.012	1.408	1.423	20′
44°50′	0.7050	0.7092	0.9942	1.006	1.410	1.418	10′
45°00′	0.7071	0.7071	1.000	1.000	1.414	1.414	45°00′
	cos	sin	cot	tan	cosec	sec	←↑

TABLE II

TRIGONOMETRIC RATIOS — DEGREES IN DECIMAL FORM

θ deg	deg	min	sin θ	cos θ	tan θ	cosec θ	sec θ	cot θ	deg	min	
0.0	0	0	0.0000	1.0000	0.0000	No value	1.0000	No value	90	0	90.0
0.1	0	6	0.0017	1.0000	0.0017	572.96	1.0000	572.96	89	54	89.9
0.2	0	12	0.0035	1.0000	0.0035	286.48	1.0000	286.48	89	48	89.8
0.3	0	18	0.0052	1.0000	0.0052	190.99	1.0000	190.98	89	42	89.7
0.4	0	24	0.0070	1.0000	0.0070	143.24	1.0000	143.24	89	36	89.6
0.5	0	30	0.0087	1.0000	0.0087	114.59	1.0000	114.59	89	30	89.5
0.6	0	36	0.0105	0.9999	0.0105	95.495	1.0001	95.490	89	24	89.4
0.7	0	42	0.0122	0.9999	0.0122	81.853	1.0001	81.847	89	18	89.3
0.8	0	48	0.0140	0.9999	0.0140	71.622	1.0001	71.615	89	12	89.2
0.9	0	54	0.0157	0.9999	0.0157	63.665	1.0001	63.657	89	6	89.1
1.0	1	0	0.0175	0.9998	0.0175	57.299	1.0002	57.290	89	0	89.0
1.1	1	6	0.0192	0.9998	0.0192	52.090	1.0002	52.081	88	54	88.9
1.2	1	12	0.0209	0.9998	0.0209	47.750	1.0002	47.740	88	48	88.8
1.3	1	18	0.0227	0.9997	0.0227	44.077	1.0003	44.066	88	42	88.7
1.4	1	24	0.0244	0.9997	0.0244	40.930	1.0003	40.917	88	36	88.6
1.5	1	30	0.0262	0.9997	0.0262	38.202	1.0003	38.188	88	30	88.5
1.6	1	36	0.0279	0.9996	0.0279	35.815	1.0004	35.801	88	24	88.4
1.7	1	42	0.0297	0.9996	0.0297	33.708	1.0004	33.694	88	18	88.3
1.8	1	48	0.0314	0.9995	0.0314	31.836	1.0005	31.821	88	12	88.2
1.9	1	54	0.0332	0.9995	0.0332	30.161	1.0005	30.145	88	6	88.1
2.0	2	0	0.0349	0.9994	0.0349	28.654	1.0006	28.636	88	0	88.0
2.1	2	6	0.0366	0.9993	0.0367	27.290	1.0007	27.271	87	54	87.9
2.2	2	12	0.0384	0.9993	0.0384	26.050	1.0007	26.031	87	48	87.8
2.3	2	18	0.0401	0.9992	0.0402	24.918	1.0008	24.898	87	42	87.7
2.4	2	24	0.0419	0.9991	0.0419	23.880	1.0009	23.859	87	36	87.6
2.5	2	30	0.0436	0.9990	0.0437	22.926	1.0010	22.904	87	30	87.5
2.6	2	36	0.0454	0.9990	0.0454	22.044	1.0010	22.022	87	24	87.4
2.7	2	42	0.0471	0.9989	0.0472	21.229	1.0011	21.205	87	18	87.3
2.8	2	48	0.0488	0.9988	0.0489	20.471	1.0012	20.446	87	12	87.2
2.9	2	54	0.0506	0.9987	0.0507	19.766	1.0013	19.740	87	6	87.1
3.0	3	0	0.0523	0.9986	0.0524	19.107	1.0014	19.081	87	0	87.0
3.1	3	6	0.0541	0.9985	0.0542	18.492	1.0015	18.464	86	54	86.9
3.2	3	12	0.0558	0.9984	0.0559	17.914	1.0016	17.886	86	48	86.8
3.3	3	18	0.0576	0.9983	0.0577	17.372	1.0017	17.343	86	42	86.7
3.4	3	24	0.0593	0.9982	0.0594	16.862	1.0018	16.832	86	36	86.6
3.5	3	30	0.0610	0.9981	0.0612	16.380	1.0019	16.350	86	30	86.5
3.6	3	36	0.0628	0.9980	0.0629	15.926	1.0020	15.895	86	24	86.4
3.7	3	42	0.0645	0.9979	0.0647	15.496	1.0021	15.464	86	18	86.3
3.8	3	48	0.0663	0.9978	0.0664	15.089	1.0022	15.056	86	12	86.2
3.9	3	54	0.0680	0.9977	0.0682	14.703	1.0023	14.669	86	6	86.1
4.0	4	0	0.0698	0.9976	0.0699	14.336	1.0024	14.301	86	0	86.0
4.1	4	6	0.0715	0.9974	0.0717	13.987	1.0026	13.951	85	54	85.9
4.2	4	12	0.0732	0.9973	0.0734	13.654	1.0027	13.617	85	48	85.8
4.3	4	18	0.0750	0.9972	0.0752	13.337	1.0028	13.300	85	42	85.7
4.4	4	24	0.0767	0.9971	0.0769	13.035	1.0030	12.996	85	36	85.6
4.5	4	30	0.0785	0.9969	0.0787	12.746	1.0031	12.706	85	30	85.5
4.6	4	36	0.0802	0.9968	0.0805	12.469	1.0032	12.429	85	24	85.4
4.7	4	42	0.0819	0.9966	0.0822	12.204	1.0034	12.163	85	18	85.3
4.8	4	48	0.0837	0.9965	0.0840	11.951	1.0035	11.909	85	12	85.2
4.9	4	54	0.0854	0.9963	0.0857	11.707	1.0037	11.665	85	6	85.1
			cos θ	sin θ	cot θ	sec θ	cosec θ	tan θ	deg	min	θ deg

TABLE II (CONTINUED)

θ deg	deg	min	sin θ	cos θ	tan θ	cosec θ	sec θ	cot θ			
5.0	5	0	0.0872	0.9962	0.0875	11.474	1.0038	11.430	85	0	85.0
5.1	5	6	0.0889	0.9960	0.0892	11.249	1.0040	11.205	84	54	84.9
5.2	5	12	0.0906	0.9959	0.0910	11.034	1.0041	10.988	84	48	84.8
5.3	5	18	0.0924	0.9957	0.0928	10.826	1.0043	10.780	84	42	84.7
5.4	5	24	0.0941	0.9956	0.0945	10.626	1.0045	10.579	84	36	84.6
5.5	5	30	0.0958	0.9954	0.0963	10.433	1.0046	10.385	84	30	84.5
5.6	5	36	0.0976	0.9952	0.0981	10.248	1.0048	10.199	84	24	84.4
5.7	5	42	0.0993	0.9951	0.0998	10.069	1.0050	10.019	84	18	84.3
5.8	5	48	0.1011	0.9949	0.1016	9.8955	1.0051	9.8448	84	12	84.2
5.9	5	54	0.1028	0.9947	0.1033	9.7283	1.0053	9.6768	84	6	84.1
6.0	6	0	0.1045	0.9945	0.1051	9.5668	1.0055	9.5144	84	0	84.0
6.1	6	6	0.1063	0.9943	0.1069	9.4105	1.0057	9.3573	83	54	83.9
6.2	6	12	0.1080	0.9942	0.1086	9.2593	1.0059	9.2052	83	48	83.8
6.3	6	18	0.1097	0.9940	0.1104	9.1129	1.0061	9.0579	83	42	83.7
6.4	6	24	0.1115	0.9938	0.1122	8.9711	1.0063	8.9152	83	36	83.6
6.5	6	30	0.1132	0.9936	0.1139	8.8337	1.0065	8.7769	83	30	83.5
6.6	6	36	0.1149	0.9934	0.1157	8.7004	1.0067	8.6428	83	24	83.4
6.7	6	42	0.1167	0.9932	0.1175	8.5711	1.0069	8.5126	83	18	83.3
6.8	6	48	0.1184	0.9930	0.1192	8.4457	1.0071	8.3863	83	12	83.2
6.9	6	54	0.1201	0.9928	0.1210	8.3238	1.0073	8.2636	83	6	83.1
7.0	7	0	0.1219	0.9925	0.1228	8.2055	1.0075	8.1444	83	0	83.0
7.1	7	6	0.1236	0.9923	0.1246	8.0905	1.0077	8.0285	82	54	82.9
7.2	7	12	0.1253	0.9921	0.1263	7.9787	1.0079	7.9158	82	48	82.8
7.3	7	18	0.1271	0.9919	0.1281	7.8700	1.0082	7.8062	82	42	82.7
7.4	7	24	0.1288	0.9917	0.1299	7.7642	1.0084	7.6996	82	36	82.6
7.5	7	30	0.1305	0.9914	0.1317	7.6613	1.0086	7.5958	82	30	82.5
7.6	7	36	0.1323	0.9912	0.1334	7.5611	1.0089	7.4947	82	24	82.4
7.7	7	42	0.1340	0.9910	0.1352	7.4635	1.0091	7.3962	82	18	82.3
7.8	7	48	0.1357	0.9907	0.1370	7.3684	1.0093	7.3002	82	12	82.2
7.9	7	54	0.1374	0.9905	0.1388	7.2757	1.0096	7.2066	82	6	82.1
8.0	8	0	0.1392	0.9903	0.1405	7.1853	1.0098	7.1154	82	0	82.0
8.1	8	6	0.1409	0.9900	0.1423	7.0972	1.0101	7.0264	81	54	81.9
8.2	8	12	0.1426	0.9898	0.1441	7.0112	1.0103	6.9395	81	48	81.8
8.3	8	18	0.1444	0.9895	0.1459	6.9273	1.0106	6.8548	81	42	81.7
8.4	8	24	0.1461	0.9893	0.1477	6.8454	1.0108	6.7720	81	36	81.6
8.5	8	30	0.1478	0.9890	0.1495	6.7655	1.0111	6.6912	81	30	81.5
8.6	8	36	0.1495	0.9888	0.1512	6.6874	1.0114	6.6122	81	24	81.4
8.7	8	42	0.1513	0.9885	0.1530	6.6111	1.0116	6.5350	81	18	81.3
8.8	8	48	0.1530	0.9882	0.1548	6.5366	1.0119	6.4596	81	12	81.2
8.9	8	54	0.1547	0.9880	0.1566	6.4637	1.0122	6.3859	81	6	81.1
9.0	9	0	0.1564	0.9877	0.1584	6.3925	1.0125	6.3138	81	0	81.0
9.1	9	6	0.1582	0.9874	0.1602	6.3228	1.0127	6.2432	80	54	80.9
9.2	9	12	0.1599	0.9871	0.1620	6.2547	1.0130	6.1742	80	48	80.8
9.3	9	18	0.1616	0.9869	0.1638	6.1880	1.0133	6.1066	80	42	80.7
9.4	9	24	0.1633	0.9866	0.1655	6.1227	1.0136	6.0405	80	36	80.6
9.5	9	30	0.1650	0.9863	0.1673	6.0589	1.0139	5.9758	80	30	80.5
9.6	9	36	0.1668	0.9860	0.1691	5.9963	1.0142	5.9124	80	24	80.4
9.7	9	42	0.1685	0.9857	0.1709	5.9351	1.0145	5.8502	80	18	80.3
9.8	9	48	0.1702	0.9854	0.1727	5.8751	1.0148	5.7894	80	12	80.2
9.9	9	54	0.1719	0.9851	0.1745	5.8164	1.0151	5.7297	80	6	80.1
			cos θ	sin θ	cot θ	sec θ	cosec θ	tan θ	deg	min	θ deg

TABLE II (CONTINUED)

θ deg	deg	min	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\operatorname{cosec}\theta$	$\sec\theta$	$\cot\theta$			
10.0	10	0	0.1736	0.9848	0.1763	5.7588	1.0154	5.6713	80	0	80.0
10.1	10	6	0.1754	0.9845	0.1781	5.7023	1.0157	5.6140	79	54	79.9
10.2	10	12	0.1771	0.9842	0.1799	5.6470	1.0161	5.5578	79	48	79.8
10.3	10	18	0.1788	0.9839	0.1817	5.5928	1.0164	5.5027	79	42	79.7
10.4	10	24	0.1805	0.9836	0.1835	5.5396	1.0167	5.4486	79	36	79.6
10.5	10	30	0.1822	0.9833	0.1853	5.4874	1.0170	5.3955	79	30	79.5
10.6	10	36	0.1840	0.9829	0.1871	5.4362	1.0174	5.3435	79	24	79.4
10.7	10	42	0.1857	0.9826	0.1890	5.3860	1.0177	5.2924	79	18	79.3
10.8	10	48	0.1874	0.9823	0.1908	5.3367	1.0180	5.2422	79	12	79.2
10.9	10	54	0.1891	0.9820	0.1926	5.2883	1.0184	5.1929	79	6	79.1
11.0	11	0	0.1908	0.9816	0.1944	5.2408	1.0187	5.1446	79	0	79.0
11.1	11	6	0.1925	0.9813	0.1962	5.1942	1:0191	5.0970	78	54	78.9
11.2	11	12	0.1942	9.9810	0.1980	5.1484	1.0194	5.0504	78	48	78.8
11.3	11	18	0.1959	0.9806	0.1998	5.1034	1.0198	5.0045	78	42	78.7
11.4	11	24	0.1977	0.9803	0.2016	5.0593	1.0201	4.9595	78	36	78.6
11.5	11	30	0.1994	0.9799	0.2035	5.0159	1.0205	4.9152	78	30	78.5
11.6	11	36	0.2011	0.9796	0.2053	4.9732	1.0209	4.8716	78	24	78.4
11.7	11	42	0.2028	0.9792	0.2071	4.9313	1.0212	4.8288	78	18	78.3
11.8	11	48	0.2045	0.9789	0.2089	4.8901	1.0216	4.7867	78	12	78.2
11.9	11	54	0.2062	0.9785	0.2107	4.8496	1.0220	4.7453	78	6	78.1
12.0	12	0	0.2079	0.9781	0.2126	4.8097	1.0223	4.7046	78	0	78.0
12.1	12	6	0.2096	0.9778	0.2144	4.7706	1.0227	4.6646	77	54	77.9
12.2	12	12	0.2113	0.9774	0.2162	4.7321	1.0231	4.6252	77	48	77.8
12.3	12	18	0.2130	0.9770	0.2180	4.6942	1.0235	4.5864	77	42	77.7
12.4	12	24	0.2147	0.9767	0.2199	4.6569	1.0239	4.5483	77	36	77.6
12.5	12	30	0.2164	0.9763	0.2217	4.6202	1.0243	4.5107	77	30	77.5
12.6	12	36	0.2181	0.9759	0.2235	4.5841	1.0247	4.4737	77	24	77.4
12.7	12	42	0.2198	0.9755	0.2254	4.5486	1.0251	4.4374	77	18	77.3
12.8	12	48	0.2215	0.9751	0.2272	4.5137	1.0255	4.4015	77	12	77.2
12.9	12	54	0.2232	0.9748	0.2290	4.4793	1.0259	4.3662	77	6	77.1
13.0	13	0	0.2250	0.9744	0.2309	4.4454	1.0263	4.3315	77	0	77.0
13.1	13	6	0.2267	0.9740	0.2327	4.4121	1.0267	4.2972	76	54	76.9
13.2	13	12	0.2284	0.9736	0.2345	4.3792	1.0271	4.2635	76	48	76.8
13.3	13	18	0.2300	0.9732	0.2364	4.3469	1.0276	4.2303	76	42	76.7
13.4	13	24	0.2317	0.9728	0.2382	4.3150	1.0280	4.1976	76	36	76.6
13.5	13	30	0.2334	0.9724	0.2401	4.2837	1.0284	4.1653	76	30	76.5
13.6	13	36	0.2351	0.9720	0.2419	4.2528	1.0288	4.1335	76	24	76.4
13.7	13	42	0.2368	0.9715	0.2438	4.2223	1.0293	4.1022	76	18	76.3
13.8	13	48	0.2385	0.9711	0.2456	4.1923	1.0297	4.0713	76	12	76.2
13.9	13	54	0.2402	0.9707	0.2475	4.1627	1.0302	4.0408	76	6	76.1
14.0	14	0	0.2419	0.9703	0.2493	4.1336	1.0306	4.0108	76	0	76.0
14.1	14	6	0.2436	0.9699	0.2512	4.1048	1.0311	3.9812	75	54	75.9
14.2	14	12	0.2453	0.9694	0.2530	4.0765	1.0315	3.9520	75	48	75.8
14.3	14	18	0.2470	0.9690	0.2549	4.0486	1.0320	3.9232	75	42	75.7
14.4	14	24	0.2487	0.9686	0.2568	4.0211	1.0324	3.8947	75	36	75.6
14.5	14	30	0.2504	0.9681	0.2586	3.9939	1.0329	3.8667	75	30	75.5
14.6	14	36	0.2521	0.9677	0.2605	3.9672	1.0334	3.8391	75	24	75.4
14.7	14	42	0.2538	0.9673	0.2623	3.9408	1.0338	3.8118	75	18	75.3
14.8	14	48	0.2554	0.9668	0.2642	3.9147	1.0343	3.7849	75	12	75.2
14.9	14	54	0.2571	0.9664	0.2661	3.8890	1.0348	3.7583	75	6	75.1
			$\cos\theta$	$\sin\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$	$\tan\theta$	deg	min	θ deg

TABLE II (CONTINUED)

θ deg	deg	min	sin θ	cos θ	tan θ	cosec θ	sec θ	cot θ			
15.0	15	0	0.2588	0.9659	0.2679	3.8637	1.0353	3.7321	75	0	75.0
15.1	15	6	0.2605	0.9655	0.2698	3.8387	1.0358	3.7062	74	54	74.9
15.2	15	12	0.2622	0.9650	0.2717	3.8140	1.0363	3.6806	74	48	74.8
15.3	15	18	0.2639	0.9646	0.2736	3.7897	1.0367	3.6554	74	42	74.7
15.4	15	24	0.2656	0.9641	0.2754	3.7657	1.0372	3.6305	74	36	74.6
15.5	15	30	0.2672	0.9636	0.2773	3.7420	1.0377	3.6059	74	30	74.5
15.6	15	36	0.2689	0.9632	0.2792	3.7186	1.0382	3.5816	74	24	74.4
15.7	15	42	0.2706	0.9627	0.2811	3.6955	1.0388	3.5576	74	18	74.3
15.8	15	48	0.2723	0.9622	0.2830	3.6727	1.0393	3.5339	74	12	74.2
15.9	15	54	0.2740	0.9617	0.2849	3.6502	1.0398	3.5105	74	6	74.1
16.0	16	0	0.2756	0.9613	0.2867	3.6280	1.0403	3.4874	74	0	74.0
16.1	16	6	0.2773	0.9608	0.2886	3.6060	1.0408	3.4646	73	54	73.9
16.2	16	12	0.2790	0.9603	0.2905	3.5843	1.0413	3.4420	73	48	73.8
16.3	16	18	0.2807	0.9598	0.2924	3.5629	1.0419	3.4197	73	42	73.7
16.4	16	24	0.2823	0.9593	0.2943	3.5418	1.0424	3.3977	73	36	73.6
16.5	16	30	0.2840	0.9588	0.2962	3.5209	1.0429	3.3759	73	30	73.5
16.6	16	36	0.2857	0.9583	0.2981	3.5003	1.0435	3.3544	73	24	74.4
16.7	16	42	0.2874	0.9578	0.3000	3.4800	1.0440	3.3332	73	18	73.3
16.8	16	48	0.2890	0.9573	0.3019	3.4598	1.0446	3.3122	73	12	73.2
16.9	16	54	0.2907	0.9568	0.3038	3.4399	1.0451	3.2914	73	6	73.1
17.0	17	0	0.2924	0.9563	0.3057	3.4203	1.0457	3.2709	73	0	73.0
17.1	17	6	0.2940	0.9558	0.3076	3.4009	1.0463	3.2506	72	54	72.9
17.2	17	12	0.2957	0.9553	0.3096	3.3817	1.0468	3.2305	72	48	72.8
17.3	17	18	0.2974	0.9548	0.3115	3.3628	1.0474	3.2106	72	42	72.7
17.4	17	24	0.2990	0.9542	0.3134	3.3440	1.0480	3.1910	72	36	72.6
17.5	17	30	0.3007	0.9537	0.3153	3.3255	1.0485	3.1716	72	30	72.5
17.6	17	36	0.3024	0.9532	0.3172	3.3072	1.0491	3.1524	72	24	72.4
17.7	17	42	0.3040	0.9527	0.3191	3.2891	1.0497	3.1334	72	18	72.3
17.8	17	48	0.3057	0.9521	0.3211	3.2712	1.0503	3.1146	72	12	72.2
17.9	17	54	0.3074	0.9516	0.3230	3.2536	1.0509	3.0961	72	6	72.1
18.0	18	0	0.3090	0.9511	0.3249	3.2361	1.0515	3.0777	72	0	72.0
18.1	18	6	0.3107	0.9505	0.3268	3.2188	1.0521	3.0595	71	54	71.9
18.2	18	12	0.3123	0.9500	0.3288	3.2017	1.0527	3.0415	71	48	71.8
18.3	18	18	0.3140	0.9494	0.3307	3.1848	1.0533	3.0237	71	42	71.7
18.4	18	24	0.3156	0.9489	0.3327	3.1681	1.0539	3.0061	71	36	71.6
18.5	18	30	0.3173	0.9483	0.3346	3.1515	1.0545	2.9887	71	30	71.5
18.6	18	36	0.3190	0.9478	0.3365	3.1352	1.0551	2.9714	71	24	71.4
18.7	18	42	0.3206	0.9472	0.3385	3.1190	1.0557	2.9544	71	18	71.3
18.8	18	48	0.3223	0.9466	0.3404	3.1030	1.0564	2.9375	71	12	71.2
18.9	18	54	0.3239	0.9461	0.3424	3.0872	1.0570	2.9208	71	6	71.1
19.0	19	0	0.3256	0.9455	0.3443	2.0716	1.0576	2.9042	71	0	71.0
19.1	19	6	0.3272	0.9449	0.3463	3.0561	1.0583	2.8878	70	54	70.9
19.2	19	12	0.3289	0.9444	0.3482	3.0407	1.0589	2.8716	70	48	70.8
19.3	19	18	0.3305	0.9438	0.3502	3.0256	1.0595	2.8556	70	42	70.7
19.4	19	24	0.3322	0.9432	0.3522	3.0106	1.0602	2.8397	70	36	70.6
19.5	19	30	0.3338	0.9426	0.3541	2.9957	1.0608	2.8239	70	30	70.5
19.6	19	36	0.3355	0.9421	0.3561	2.9811	1.0615	2.8083	70	24	70.4
19.7	19	42	0.3371	0.9415	0.3581	2.9665	1.0622	2.7929	70	18	70.3
19.8	19	48	0.3387	0.9409	0.3600	2.9521	1.0628	2.7776	70	12	70.2
19.9	19	54	0.3404	0.9403	0.3620	2.9379	1.0635	2.7625	70	6	70.1
			cos θ	sin θ	cot θ	sec θ	cosec θ	tan θ	deg	min	θ deg

TABLE II (CONTINUED)

θ deg	deg	min	$\sin\theta$	$\cos\theta$	$\tan\theta$	cosec θ	$\sec\theta$	$\cot\theta$			
20.0	20	0	0.3420	0.9397	0.3640	2.9238	1.0642	2.7475	70	0	70.0
20.1	20	6	0.3437	0.9391	0.3659	2.9099	1.0649	2.7326	69	54	69.9
20.2	20	12	0.3453	0.9385	0.3679	2.8960	1.0655	2.7179	69	48	69.8
20.3	20	18	0.3469	0.9379	0.3699	2.8824	1.0662	2.7034	69	42	69.7
20.4	20	24	0.3486	0.9373	0.3719	2.8688	1.0669	2.6889	69	36	69.6
20.5	20	30	0.3502	0.9367	0.3739	2.8555	1.0676	2.6746	69	30	69.5
20.6	20	36	0.3518	0.9361	0.3759	2.8422	1.0683	2.6605	69	24	69.4
20.7	20	42	0.3535	0.9354	0.3779	2.8291	1.0690	2.6464	69	18	69.3
20.8	20	48	0.3551	0.9348	0.3799	2.8161	1.0697	2.6325	69	12	69.2
20.9	20	54	0.3567	0.9342	0.3819	2.8032	1.0704	2.6187	69	6	69.1
21.0	21	0	0.3584	0.9336	0.3839	2.7904	1.0711	2.6051	69	0	69.0
21.1	21	6	0.3600	0.9330	0.3859	2.7778	1.0719	2.5916	68	54	68.9
21.2	21	12	0.3616	0.9323	0.3879	2.7653	1.0726	2.5782	68	48	68.8
21.3	21	18	0.3633	0.9317	0.3899	2.7529	1.0733	2.5649	68	42	68.7
21.4	21	24	0.3649	0.9311	0.3919	2.7407	1.0740	2.5517	68	36	68.6
21.5	21	30	0.3665	0.9304	0.3939	2.7285	1.0748	2.5386	68	30	68.5
21.6	21	36	0.3681	0.9298	0.3959	2.7165	1.0755	2.5257	68	24	68.4
21.7	21	42	0.3697	0.9291	0.3979	2.7046	1.0763	2.5129	68	18	68.3
21.8	21	48	0.3714	0.9285	0.4000	2.6927	1.0770	2.5002	68	12	68.2
21.9	21	54	0.3730	0.9278	0.4020	2.6811	1.0778	2.4876	68	6	68.1
22.0	22	0	0.3746	0.9272	0.4040	2.6695	1.0785	2.4751	68	0	68.0
22.1	22	6	0.3762	0.9265	0.4061	2.6580	1.0793	2.4627	67	54	67.9
22.2	22	12	0.3778	0.9259	0.4081	2.6466	1.0801	2.4504	67	48	67.8
22.3	22	18	0.3795	0.9252	0.4101	2.6354	1.0808	2.4383	67	42	67.7
22.4	22	24	0.3811	0.9245	0.4122	2.6242	1.0816	2.4262	67	36	67.6
22.5	22	30	0.3827	0.9239	0.4142	2.6131	1.0824	2.4142	67	30	67.5
22.6	22	36	0.3843	0.9232	0.4163	2.6022	1.0832	2.4023	67	24	67.4
22.7	22	42	0.3859	0.9225	0.4183	2.5913	1.0840	2.3906	67	18	67.3
22.8	22	48	0.3875	0.9219	0.4204	2.5805	1.0848	2.3789	67	12	67.2
22.9	22	54	0.3891	0.9212	0.4224	2.5699	1.0856	2.3673	67	6	67.1
23.0	23	0	0.3907	0.9205	0.4245	2.5593	1.0864	2.3559	67	0	67.0
23.1	23	6	0.3923	0.9198	0.4265	2.5488	1.0872	2.3445	66	54	66.9
23.2	23	12	0.3939	0.9191	0.4286	2.5384	1.0880	2.3332	66	48	66.8
23.3	23	18	0.3955	0.9184	0.4307	2.5282	1.0888	2.3220	66	42	66.7
23.4	23	24	0.3971	0.9178	0.4327	2.5180	1.0896	2.3109	66	36	66.6
23.5	23	30	0.3987	0.9171	0.4348	2.5078	1.0904	2.2998	66	30	66.5
23.6	23	36	0.4003	0.9164	0.4369	2.4978	1.0913	2.2889	66	24	66.4
23.7	23	42	0.4019	0.9157	0.4390	2.4879	1.0921	2.2781	66	18	66.3
23.8	23	48	0.4035	0.9150	0.4411	2.4780	1.0929	2.2673	66	12	66.2
23.9	23	54	0.4051	0.9143	0.4431	2.4683	1.0938	2.2566	66	6	66.1
24.0	24	0	0.4067	0.9135	0.4452	2.4586	1.0946	2.2460	66	0	66.0
24.1	24	6	0.4083	0.9128	0.4473	2.4490	1.0955	2.2355	65	54	65.9
24.2	24	12	0.4099	0.9121	0.4494	2.4395	1.0963	2.2251	65	48	65.8
24.3	24	18	0.4115	0.9114	0.4515	2.4301	1.0972	2.2148	65	42	65.7
24.4	24	24	0.4131	0.9107	0.4536	2.4207	1.0981	2.2045	65	36	65.6
24.5	24	30	0.4147	0.9100	0.4557	2.4114	1.0989	2.1943	65	30	65.5
24.6	24	36	0.4163	0.9092	0.4578	2.4022	1.0998	2.1842	65	24	65.4
24.7	24	42	0.4179	0.9085	0.4599	2.3931	1.1007	2.1742	65	18	65.3
24.8	24	48	0.4195	0.9078	0.4621	2.3841	1.1016	2.1642	65	12	65.2
24.9	24	54	0.4210	0.9070	0.4642	2.3751	1.1025	2.1543	65	6	65.1
			$\cos\theta$	$\sin\theta$	$\cot\theta$	$\sec\theta$	cosec θ	$\tan\theta$	deg	min	θ deg

TABLE II (CONTINUED)

θ deg	deg min	sin θ	cos θ	tan θ	cosec θ	sec θ	cot θ		
25.0	25 0	0.4226	0.9063	0.4663	2.3662	1.1034	2.1445	65 0	65.0
25.1	25 6	0.4242	0.9056	0.4684	2.3574	1.1043	2.1348	64 54	64.9
25.2	25 12	0.4258	0.9048	0.4706	2.3486	1.1052	2.1251	64 48	64.8
25.3	25 18	0.4274	0.9041	0.4727	2.3400	1.1061	2.1155	64 42	64.7
25.4	25 24	0.4289	0.9033	0.4748	2.3314	1.1070	2.1060	64 36	64.6
25.5	25 30	0.4305	0.9026	0.4770	2.3228	1.1079	2.0965	64 30	64.5
25.6	25 36	0.4321	0.9018	0.4791	2.3144	1.1089	2.0872	64 24	64.4
25.7	25 42	0.4337	0.9011	0.4813	2.3060	1.1098	2.0778	64 18	64.3
25.8	25 48	0.4352	0.9003	0.4834	2.2976	1.1107	2.0686	64 12	64.2
25.9	25 54	0.4368	0.8996	0.4856	2.2894	1.1117	2.0594	64 6	64.1
26.0	26 0	0.4384	0.8988	0.4877	2.2812	1.1126	2.0503	64 0	64.0
26.1	26 6	0.4399	0.8980	0.4899	2.2730	1.1136	2.0413	63 54	63.9
26.2	26 12	0.4415	0.8973	0.4921	2.2650	1.1145	2.0323	63 48	63.8
26.3	26 18	0.4431	0.8965	0.4942	2.2570	1.1155	2.0233	63 42	63.7
26.4	26 24	0.4446	0.8957	0.4964	2.2490	1.1164	2.0145	63 36	63.6
26.5	26 30	0.4462	0.8949	0.4986	2.2412	1.1174	2.0057	63 30	63.5
26.6	26 36	0.4478	0.8942	0.5008	2.2333	1.1184	1.9970	63 24	63.4
26.7	26 42	0.4493	0.8934	0.5029	2.2256	1.1194	1.9883	63 18	63.3
26.8	26 48	0.4509	0.8926	0.5051	2.2179	1.1203	1.9797	63 12	63.2
26.9	26 54	0.4524	0.8918	0.5073	2.2103	1.1213	1.9711	63 6	63.1
27.0	27 0	0.4540	0.8910	0.5095	2.2027	1.1223	1.9626	63 0	63.0
27.1	27 6	0.4555	0.8902	0.5117	2.1952	1.1233	1.9542	62 54	62.9
27.2	27 12	0.4571	0.8894	0.5139	2.1877	1.1243	1.9458	62 48	62.8
27.3	27 18	0.4586	0.8886	0.5161	2.1803	1.1253	1.9375	62 42	62.7
27.4	27 24	0.4602	0.8878	0.5184	2.1730	1.1264	1.9292	62 36	62.6
27.5	27 30	0.4617	0.8870	0.5206	2.1657	1.1274	1.9210	62 30	62.5
27.6	27 36	0.4633	0.8862	0.5228	2.1584	1.1284	1.9128	62 24	62.4
27.7	27 42	0.4648	0.8854	0.5250	2.1513	1.1294	1.9047	62 18	62.3
27.8	27 48	0.4664	0.8846	0.5272	2.1441	1.1305	1.8967	62 12	62.2
27.9	27 54	0.4679	0.8838	0.5295	2.1371	1.1315	1.8887	62 6	62.1
28.0	28 0	0.4695	0.8829	0.5317	2.1301	1.1326	1.8807	62 0	62.0
28.1	28 6	0.4710	0.8821	0.5339	2.1231	1.1336	1.8728	61 54	61.9
28.2	28 12	0.4726	0.8813	0.5362	2.1162	1.1347	1.8650	61 48	61.8
28.3	28 18	0.4741	0.8805	0.5384	2.1093	1.1357	1.8572	61 42	61.7
28.4	28 24	0.4756	0.8796	0.5407	2.1025	1.1368	1.8495	61 36	61.6
28.5	28 30	0.4772	0.8788	0.5430	2.0957	1.1379	1.8418	61 30	61.5
28.6	28 36	0.4787	0.8780	0.5452	2.0890	1.1390	1.8341	61 24	61.4
28.7	28 42	0.4802	0.8771	0.5475	2.0824	1.1401	1.8265	61 18	61.3
28.8	28 48	0.4818	0.8763	0.5498	2.0758	1.1412	1.8190	61 12	61.2
28.9	28 54	0.4833	0.8755	0.5520	2.0692	1.1423	1.8115	61 6	61.1
29.0	29 0	0.4848	0.8746	0.5543	2.0627	1.1434	1.8040	61 0	61.0
29.1	29 6	0.4863	0.8738	0.5566	2.0562	1.1445	1.7966	60 54	60.9
29.2	29 12	0.4879	0.8729	0.5589	2.0598	1.1456	1.7893	60 48	60.8
29.3	29 18	0.4894	0.8721	0.5612	2.0434	1.1467	1.7820	60 42	60.7
29.4	29 24	0.4909	0.8712	0.5635	2.0371	1.1478	1.7747	60 36	60.6
29.5	29 30	0.4924	0.8704	0.5658	2.0308	1.1490	1.7675	60 30	60.5
29.6	29 36	0.4939	0.8695	0.5681	2.0245	1.1501	1.7603	60 24	60.4
29.7	29 42	0.4955	0.8686	0.5704	2.0183	1.1512	1.7532	60 18	60.3
29.8	29 48	0.4970	0.8678	0.5727	2.0122	1.1524	1.7461	60 12	60.2
29.9	29 54	0.4985	0.8669	0.5750	2.0061	1.1535	1.7391	60 6	60.1
		cos θ	sin θ	cot θ	sec θ	cosec θ	tan θ	deg min	θ deg

TABLE II (CONTINUED)

θ deg	deg	min	sin θ	cos θ	tan θ	cosec θ	sec θ	cot θ	deg	min	θ deg
30.0	30	0	0.5000	0.8660	0.5774	2.0000	1.1547	1.7321	60	0	60.0
30.1	30	6	0.5015	0.8652	0.5797	1.9940	1.1559	1.7251	59	54	59.9
30.2	30	12	0.5030	0.8643	0.5820	1.9880	1.1570	1.7182	59	48	59.8
30.3	30	18	0.5045	0.8634	0.5844	1.9821	1.1582	1.7113	59	42	59.7
30.4	30	24	0.5060	0.8625	0.5867	1.9762	1.1594	1.7045	59	36	59.6
30.5	30	30	0.5075	0.8616	0.5890	1.9703	1.1606	1.6977	59	30	59.5
30.6	30	36	0.5090	0.8607	0.5914	1.9645	1.1618	1.6909	59	24	59.4
30.7	30	42	0.5105	0.8599	0.5938	1.9587	1.1630	1.6842	59	18	59.3
30.8	30	48	0.5120	0.8590	0.5961	1.9530	1.1642	1.6775	59	12	59.2
30.9	30	54	0.5135	0.8581	0.5985	1.9473	1.1654	1.6709	59	6	59.1
31.0	31	0	0.5150	0.8572	0.6009	1.9416	1.1666	1.6643	59	0	59.0
31.1	31	6	0.5165	0.8563	0.6032	1.9360	1.1679	1.6577	58	54	58.9
31.2	31	12	0.5180	0.8554	0.6056	1.9304	1.1691	1.6512	58	48	58.8
31.3	31	18	0.5195	0.8545	0.6080	1.9249	1.1703	1.6447	58	42	58.7
31.4	31	24	0.5210	0.8536	0.6104	1.9194	1.1716	1.6383	58	36	58.6
31.5	31	30	0.5225	0.8526	0.6128	1.9139	1.1728	1.6319	58	30	58.5
31.6	31	36	0.5240	0.8517	0.6152	1.9084	1.1741	1.6255	58	24	58.4
31.7	31	42	0.5255	0.8508	0.6176	1.9031	1.1753	1.6191	58	18	58.3
31.8	31	48	0.5270	0.8499	0.6200	1.8977	1.1766	1.6128	58	12	58.2
31.9	31	54	0.5284	0.8490	0.6224	1.8924	1.1779	1.6066	58	6	58.1
32.0	32	0	0.5299	0.8480	0.6249	1.8871	1.1792	1.6003	58	0	58.0
32.1	32	6	0.5314	0.8471	0.6273	1.8818	1.1805	1.5941	57	54	57.9
32.2	32	12	0.5329	0.8462	0.6297	1.8766	1.1818	1.5880	57	48	57.8
32.3	32	18	0.5344	0.8453	0.6322	1.8714	1.1831	1.5818	57	42	57.7
32.4	32	24	0.5358	0.8443	0.6346	1.8663	1.1844	1.5757	57	36	57.6
32.5	32	30	0.5373	0.8434	0.6371	1.8612	1.1857	1.5697	57	30	57.5
32.6	32	36	0.5388	0.8425	0.6395	1.8561	1.1870	1.5637	57	24	57.4
32.7	32	42	0.5402	0.8415	0.6420	1.8510	1.1883	1.5577	57	18	57.3
32.8	32	48	0.5417	0.8406	0.6445	1.8460	1.1897	1.5517	57	12	57.2
32.9	32	54	0.5432	0.8396	0.6469	1.8410	1.1910	1.5458	57	6	57.1
33.0	33	0	0.5446	0.8387	0.6494	1.8361	1.1924	1.5399	57	0	57.0
33.1	33	6	0.5461	0.8377	0.6519	1.8312	1.1937	1.5340	56	54	56.9
33.2	33	12	0.5476	0.8368	0.6544	1.8263	1.1951	1.5282	56	48	56.8
33.3	33	18	0.5490	0.8358	0.6569	1.8214	1.1964	1.5224	56	42	56.7
33.4	33	24	0.5505	0.8348	0.6594	1.8166	1.1978	1.5166	56	36	56.6
33.5	33	30	0.5519	0.8339	0.6619	1.8118	1.1992	1.5108	56	30	56.5
33.6	33	36	0.5534	0.8329	0.6644	1.8070	1.2006	1.5051	56	24	56.4
33.7	33	42	0.5548	0.8320	0.6669	1.8023	1.2020	1.4994	56	18	56.3
33.8	33	48	0.5563	0.8310	0.6694	1.7976	1.2034	1.4938	56	12	56.2
33.9	33	54	0.5577	0.8300	0.6720	1.7929	1.2048	1.4882	56	6	56.1
34.0	34	0	0.5592	0.8290	0.6745	1.7883	1.2062	1.4826	56	0	56.0
34.1	34	6	0.5606	0.8281	0.6771	1.7837	1.2076	1.4770	55	54	55.9
34.2	34	12	0.5621	0.8271	0.6796	1.7791	1.2091	1.4715	55	48	55.8
34.3	34	18	0.5635	0.8261	0.6822	1.7745	1.2105	1.4659	55	42	55.7
34.4	34	24	0.5650	0.8251	0.6847	1.7700	1.2120	1.4605	55	36	55.6
34.5	34	30	0.5664	0.8241	0.6873	1.7655	1.2134	1.4550	55	30	55.5
34.6	34	36	0.5678	0.8231	0.6899	1.7610	1.2149	1.4496	55	24	55.4
34.7	34	42	0.5693	0.8221	0.6924	1.7566	1.2163	1.4442	55	18	55.3
34.8	34	48	0.5707	0.8211	0.6950	1.7522	1.2178	1.4388	55	12	55.2
34.9	34	54	0.5721	0.8202	0.6976	1.7478	1.2193	1.4335	55	6	55.1
			cos θ	sin θ	cot θ	sec θ	cosec θ	tan θ	deg	min	θ deg

TABLE II (CONTINUED)

θ deg	deg	min	sin θ	cos θ	tan θ	cosec θ	sec θ	cot θ	deg	min	θ deg
35.0	35	0	0.5736	0.8192	0.7002	1.7434	1.2208	1.4281	55	0	55.0
35.1	35	6	0.5750	0.8181	0.7028	1.7391	1.2223	1.4229	54	54	54.9
35.2	35	12	0.5764	0.8171	0.7054	1.7348	1.2238	1.4176	54	48	54.8
35.3	35	18	0.5779	0.8161	0.7080	1.7305	1.2253	1.4124	54	42	54.7
35.4	35	24	0.5793	0.8151	0.7107	1.7263	1.2268	1.4071	54	36	54.6
35.5	35	30	0.5807	0.8141	0.7133	1.7221	1.2283	1.4019	54	30	54.5
35.6	35	36	0.5821	0.8131	0.7159	1.7179	1.2299	1.3968	54	24	54.4
35.7	35	42	0.5835	0.8121	0.7186	1.7137	1.2314	1.3916	54	18	54.3
35.8	35	48	0.5850	0.8111	0.7212	1.7095	1.2329	1.3865	54	12	54.2
35.9	35	54	0.5864	0.8100	0.7239	1.7054	1.2345	1.3814	54	6	54.1
36.0	36	0	0.5878	0.8090	0.7265	1.7013	1.2361	1.3764	54	0	54.0
36.1	36	6	0.5892	0.8080	0.7292	1.6972	1.2376	1.3713	53	54	53.9
36.2	36	12	0.5906	0.8070	0.7319	1.6932	1.2392	1.3663	53	48	53.8
36.3	36	18	0.5920	0.8059	0.7346	1.6892	1.2408	1.3613	53	42	53.7
36.4	36	24	0.5934	0.8049	0.7373	1.6852	1.2424	1.3564	53	36	53.6
36.5	36	30	0.5948	0.8039	0.7400	1.6812	1.2440	1.3514	53	30	53.5
36.6	36	36	0.5962	0.8028	0.7427	1.6772	1.2456	1.3465	53	24	53.4
36.7	36	42	0.5976	0.8018	0.7454	1.6733	1.2472	1.3416	53	18	53.3
36.8	36	48	0.5990	0.8007	0.7481	1.6694	1.2489	1.3367	53	12	53.2
36.9	36	54	0.6004	0.7997	0.7508	1.6655	1.2505	1.3319	53	6	53.1
37.0	37	0	0.6018	0.7986	0.7536	1.6616	1.2521	1.3270	53	0	53.0
37.1	37	6	0.6032	0.7976	0.7563	1.6578	1.2538	1.3222	52	54	52.9
37.2	37	12	0.6046	0.7965	0.7590	1.6540	1.2554	1.3175	52	48	52.8
37.3	37	18	0.6060	0.7955	0.7618	1.6502	1.2571	1.3127	52	42	52.7
37.4	37	24	0.6074	0.7944	0.7646	1.6464	1.2588	1.3079	52	36	52.6
37.5	37	30	0.6088	0.7934	0.7673	1.6427	1.2605	1.3032	52	30	52.5
37.6	37	36	0.6101	0.7923	0.7701	1.6390	1.2622	1.2985	52	24	52.4
37.7	37	42	0.6115	0.7912	0.7729	1.6353	1.2639	1.2938	52	18	52.3
37.8	37	48	0.6129	0.7902	0.7757	1.6316	1.2656	1.2892	52	12	52.2
37.9	37	54	0.6143	0.7891	0.7785	1.6279	1.2673	1.2846	52	6	52.1
38.0	38	0	0.6157	0.7880	0.7813	1.6243	1.2690	1.2799	52	0	52.0
38.1	38	6	0.6170	0.7869	0.7841	1.6207	1.2708	1.2753	51	54	51.9
38.2	38	12	0.6184	0.7859	0.7869	1.6171	1.2725	1.2708	51	48	51.8
38.3	38	18	0.6198	0.7848	0.7898	1.6135	1.2742	1.2662	51	42	51.7
38.4	38	24	0.6211	0.7837	0.7926	1.6099	1.2760	1.2617	51	36	51.6
38.5	38	30	0.6225	0.7826	0.7954	1.6064	1.2778	1.2572	51	30	51.5
38.6	38	36	0.6239	0.7815	0.7983	1.6029	1.2796	1.2527	51	24	51.4
38.7	38	42	0.6252	0.7804	0.8012	1.5994	1.2813	1.2482	51	18	51.3
38.8	38	48	0.6266	0.7793	0.8040	1.5959	1.2831	1.2437	51	12	51.2
38.9	38	54	0.6280	0.7782	0.8069	1.5925	1.2849	1.2393	51	6	51.1
39.0	39	0	0.6293	0.7771	0.8098	1.5890	1.2868	1.2349	51	0	51.0
39.1	39	6	0.6307	0.7760	0.8127	1.5856	1.2886	1.2305	50	54	50.9
39.2	39	12	0.6320	0.7749	0.8156	1.5822	1.2904	1.2261	50	48	50.8
39.3	39	18	0.6334	0.7738	0.8185	1.5788	1.2923	1.2218	50	42	50.7
39.4	39	24	0.6347	0.7727	0.8214	1.5755	1.2941	1.2174	50	36	50.6
39.5	39	30	0.6361	0.7716	0.8243	1.5721	1.2960	1.2131	50	30	50.5
39.6	39	36	0.6374	0.7705	0.8273	1.5688	1.2978	1.2088	50	24	50.4
39.7	39	42	0.6388	0.7694	0.8302	1.5655	1.2997	1.2045	50	18	50.3
39.8	39	48	0.6401	0.7683	0.8332	1.5622	1.3016	1.2002	50	12	50.2
39.9	39	54	0.6414	0.7672	0.8361	1.5590	1.3035	1.1960	50	6	50.1
			cos θ	sin θ	cot θ	sec θ	cosec θ	tan θ	deg	min	θ deg

TABLE II (CONTINUED)

θ deg	deg min	sin θ	cos θ	tan θ	cosec θ	sec θ	cot θ		
40.0	40 0	0.6428	0.7660	0.8391	1.5557	1.3054	1.1918	50 0	50.
40.1	40 6	0.6441	0.7649	0.8421	1.5525	1.3073	1.1875	49 54	49.
40.2	40 12	0.6455	0.7638	0.8451	1.5493	1.3092	1.1833	49 48	49.
40.3	40 18	0.6468	0.7627	0.8481	1.5461	1.3112	1.1792	49 42	49.7
40.4	40 24	0.6481	0.7615	0.8511	1.5429	1.3131	1.1750	49 36	49.6
40.5	40 30	0.6494	0.7604	0.8541	1.5398	1.3151	1.1708	49 30	49.5
40.6	40 36	0.6508	0.7593	0.8571	1.5366	1.3171	1.1667	49 24	49.4
40.7	40 42	0.6521	0.7581	0.8601	1.5335	1.3190	1.1626	49 18	49.3
40.8	40 48	0.6534	0.7570	0.8632	1.5304	1.3210	1.1585	49 12	49.2
40.9	40 54	0.6547	0.7559	0.8662	1.5273	1.3230	1.1544	49 6	49.1
41.0	41 0	0.6561	0.7547	0.8693	1.5243	1.3250	1.1504	49 0	49.0
41.1	41 6	0.6574	0.7536	0.8724	1.5212	1.3270	1.1463	48 54	48.9
41.2	41 12	0.6587	0.7524	0.8754	1.5182	1.3291	1.1423	48 48	48.8
41.3	41 18	0.6600	0.7513	0.8785	1.5151	1.3311	1.1383	48 42	48.7
41.4	41 24	0.6613	0.7501	0.8816	1.5121	1.3331	1.1343	48 36	48.6
41.5	41 30	0.6626	0.7490	0.8847	1.5092	1.3352	1.1303	48 30	48.5
41.6	41 36	0.6639	0.7478	0.8878	1.5062	1.3373	1.1263	48 24	48.4
41.7	41 42	0.6652	0.7466	0.8910	1.5032	1.3393	1.1224	48 18	48.3
41.8	41 48	0.6665	0.7455	0.8941	1.5003	1.3414	1.1184	48 12	48.2
41.9	41 54	0.6678	0.7443	0.8972	1.4974	1.3435	1.1145	48 6	48.1
42.0	42 0	0.6691	0.7431	0.9004	1.4945	1.3456	1.1106	48 0	48.
42.1	42 6	0.6704	0.7420	0.9036	1.4916	1.3478	1.1067	47 54	47.9
42.2	42 12	0.6717	0.7408	0.9067	1.4887	1.3499	1.1028	47 48	47.8
42.3	42 18	0.6730	0.7396	0.9099	1.4859	1.3520	1.0990	47 42	47.
42.4	42 24	0.6743	0.7385	0.9131	1.4830	1.3542	1.0951	47 36	47.
42.5	42 30	0.6756	0.7373	0.9163	1.4802	1.3563	1.0913	47 30	47.
42.6	42 36	0.6769	0.7361	0.9195	1.4774	1.3585	1.0875	47 24	47
42.7	42 42	0.6782	0.7349	0.9228	1.4746	1.3607	1.0837	47 18	47
42.8	42 48	0.6794	0.7337	0.9260	1.4718	1.3629	1.0799	47 12	47.
42.9	42 54	0.6807	0.7325	0.9293	1.4690	1.3651	1.0761	47 6	47.
43.0	43 0	0.6820	0.7314	0.9325	1.4663	1.3673	1.0724	47 0	47.
43.1	43 6	0.6833	0.7302	0.9358	1.4635	1.3696	1.0686	46 54	46
43.2	43 12	0.6845	0.7290	0.9391	1.4608	1.3718	1.0649	46 48	46
43.3	43 18	0.6858	0.7278	0.9424	1.4581	1.3741	1.0612	46 42	46
43.4	43 24	0.6871	0.7266	0.9457	1.4554	1.3763	1.0575	46 36	46.
43.5	43 30	0.6884	0.7254	0.9490	1.4527	1.3786	1.0538	46 30	46.
43.6	43 36	0.6896	0.7242	0.9523	1.4501	1.3809	1.0501	46 24	46.
43.7	43 42	0.6909	0.7230	0.9556	1.4474	1.3832	1.0464	46 18	46.
43.8	43 48	0.6921	0.7218	0.9590	1.4448	1.3855	1.0428	46 12	46
43.9	43 54	0.6934	0.7206	0.9623	1.4422	1.3878	1.0392	46 6	46.
44.0	44 0	0.6947	0.7193	0.9657	1.4396	1.3902	1.0355	46 0	46.
44.1	44 6	0.6959	0.7181	0.9691	1.4370	1.3925	1.0319	45 54	45
44.2	44 12	0.6972	0.7169	0.9725	1.4344	1.3949	1.0283	45 48	45
44.3	44 18	0.6984	0.7157	0.9759	1.4318	1.3972	1.0247	45 42	45
44.4	44 24	0.6997	0.7145	0.9793	1.4293	1.3996	1.0212	45 36	45
44.5	44 30	0.7009	0.7133	0.9827	1.4267	1.4020	1.0176	45 30	45.
44.6	44 36	0.7022	0.7120	0.9861	1.4242	1.4044	1.0141	45 24	45.4
44.7	44 42	0.7034	0.7108	0.9896	1.4217	1.4069	1.0105	45 18	45.3
44.8	44 48	0.7046	0.7096	0.9930	1.4192	1.4093	1.0070	45 12	45
44.9	44 54	0.7059	0.7083	0.9965	1.4167	1.4118	1.0035	45 6	45.1
45.0	45 0	0.7071	0.7071	1.0000	1.4142	1.4142	1.0000	45 0	45.
		cos θ	sin θ	cot θ	sec θ	cosec θ	tan θ	deg min	θ deg